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OUTER SOLUTIONS FOR GENERAL LINEAR TURNING POINT PROBLEMS.(U)

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Wolfgang Wasow

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Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

In the differential equation

$$\varepsilon^h \frac{dy}{dx} = A(x, \varepsilon)y$$

let $A(x, \varepsilon)$ be an $n \times n$ matrix valued function of the complex variable x and the parameter ε, holomorphic in both variables for $|x| \leq x_0$, $0 < \varepsilon \leq \varepsilon_0$. Let y be a vector and h a positive integer.

Fundamental matrix solutions of the differential equation are constructed which involve functions that can be represented asymptotically by series in powers of ε. The domains of validity of these expansions grow, as ε tends to zero, in such a way that their distance from $x = 0$ tends to zero with ε. Combined with stretching transformations of the form $x = \xi\varepsilon^\rho$, this result is a basic tool for the analysis of turning points of unrestricted complexity.

Equivalent results are contained in papers by M. Iwano (Items [6], [7], [8] of the bibliography at the end of this paper.) The method of the present article has many points of similarity with Iwano's work, but it is substantially simpler and shorter. This is achieved by a systematic use of a somewhat generalized concept of asymptotic series in powers of ε for functions of x and ε, by drawing on results by H. Turrittin and V. I. Arnold, and by avoiding the preparatory reduction of the problem to a block-triangular form, necessary in Iwano's presentation.

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OUTER SOLUTIONS FOR GENERAL LINEAR TURNING POINT PROBLEMS

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§ 1. Introduction.

There is a vast body of literature on the properties of linear analytic differential equations near a pole with respect to a parameter. For general investigations such differential equations are usually written in the form

$$\varepsilon^h \frac{dy}{dx} = A(x, \varepsilon)y, \tag{1.1}$$

where $A(x, \varepsilon)$ is an $n \times n$ matrix-valued function, y an n-dimensional column vector, ε a small parameter and h a positive integer.

The main difficulty in these theories stems from the fact that the algebraically relevant properties of the coefficient matrix $A(x, \varepsilon)$ may change discontinuously, even at points where it is holomorphic. Some restrictions on $A(x, \varepsilon)$ have therefore been imposed in most papers in this field, whether they deal with local or global questions.

M. Iwano, in three connected papers [6], [7], [8], has developed a theory that is truly general, in that for any problem of the form (1.1) a method is described by which, for any point $x = x_0$, a set of asymptotic series for certain fundamental solutions can be calculated so that the domains of validity of these expansions together cover a whole neighborhood

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of x_0. The great length and complexity of these investigations makes it desirable to simplify and shorten the presentation, so as to make it more easily accessible. This is the aim of the present paper. Its arguments have many points of similarity witn Iwano's, but there are also some important differences. In particular, I permit from the beginning matrices that are not block-triangular, a restriction which Iwano removes only in the Appendix of his third paper. The widened concept of asymptotic power series introduced in section 2 is a helpful tool which shortens the subsequent analysis. In addition, the presentation is facilitated by employing certain results of Turrittin [17] and of Arnold [1].

Near points of the complex x-plane where all eigenvalues of $A(x,0)$ are distinct, a complete solution of (1.1) in terms of asymptotic series in powers of ε has been known since early in this century. Its description can be found in several textbooks, e.g., [2] or [18].

The existing knowledge near points where $A(x,0)$ has multiple eigenvalues is still incomplete. If $A(x,0)$ has no eigenvalues which are identically equal, there exists a set of isolated points in the x-plane at which multiple eigenvalues occur. They are often called "turning points". The asymptotic nature of the solutions near such points can be extraordinarily complicated (see, e.g., [9], [10], [11]) and involves as yet unsolved connection problems. If $A(x,0)$ possesses some eigenvalues that are identically equal, it is not even clear what terminology should be adopted. Is it then reasonable to call every point in the domain

of $A(x, 0)$ a turning point? One might be tempted to apply that term only to points, where some eigenvalues are equal without being identically equal. However, as Hukuhara [5] and Turrittin [17] have shown, the asymptotic analysis near points where all multiple roots are identically equal often leads unavoidably to problems that have there a turning point in the former sense.

I shall, therefore, not attempt to offer a mathematically precise definition of the concept of "turning point", but I will use these words in a descriptive sense as having to do with difficulties in the asymptotic theory arising near certain points from abrupt changes in the algebraic structure of relevant matrices at that point.

The most generally applicable method of approach to turning point problems is based on "stretching" and "matching". If the point in question is at $x = 0$, a transformation of the form $x = \xi\varepsilon^{\kappa}$, $\kappa > 0$, is fittingly called a stretching. For properly chosen values of κ the "stretched" differential equation can often be solved or, at least, brought closer to a solution, in bounded domains of the ξ-plane. As the image in the x-plane of such a domain shrinks to zero with ε, a successful "matching" requires that, in addition, asymptotic solutions are available in regions which, while possibly bounded away from the turning point, expand towards it, as ε tends to zero. The two types of solutions are often called "inner" and "outer" solutions.

As Iwano and Sibuya have shown ([6], [7], [8], [10]) the stretched problem may again require a composite analysis, involving series of "outer" and "inner" type with respect to the new, stretched, variable. The ingenious and methodical work in the papers cited above does not yet solve the problem of the connection of the several fundamental solutions with known asymptotic series valid in different domains. For partial results concerning the calculation of the "connection matrices" see [14] and [21]. As far as the calculation of each of these solutions is concerned, all the features appear already in the outer solution for the original differential equation. A few further remarks on this point can be found at the end of section 11.

Because of the length of the arguments it is desirable to begin with a description of the main result in a language that is not excessively technical:

The asymptotic solution of the differential equation (1.1) constructed in this paper involves asymptotic series in powers of ε whose coefficients are not necessarily holomorphic in x at the point $x = 0$ near which the outer solutions are to be found. These coefficients may grow as fast as some negative powers of x, as the point $x = 0$ is approached. However, the exponents of these negative powers do not increase faster than at a linear rate with the order of the term in the series. Thus, if $f_r(x)\varepsilon^r$ is the r^{th} term of such a series, there exist two numbers, p and κ, independent of r such that $f_r(x)x^{p+r/\kappa}$ remains bounded as

$x \to 0$, at least in some sector of the x-plane. Functions of x and ε that possess such expansions are said to be in class $\mathcal{A}^{**}$. For precise definitions read Definitions 2.1 and 9.1 below.

Theorem 1.1 (Main Theorem). Let

$$\varepsilon^h \frac{dy}{dx} = A(x, \varepsilon)y; \; h > 0, \quad \text{integer}, \tag{1.1}$$

be a system of n scalar differential equations whose coefficient matrix is holomorphic in $|x| \le x_0$, $0 < \varepsilon \le \varepsilon_0$ and possesses an asymptotic expansion

$$A(x, \varepsilon) \sim \sum_{r=0}^{\infty} A_r(x)\varepsilon^r, \quad \varepsilon \to 0+,$$

(1.2) uniformly in $|\arg x| \le \delta_0$. Then (1.1) has a fundamental matrix solution $Y(x, \varepsilon)$ of the form

$$Y(x, \varepsilon) = \hat{Y}(\tilde{x}, \tilde{\varepsilon})\, e^{\tilde{\varepsilon}^{-H} Q(\tilde{x}, \tilde{\varepsilon})} \tag{1.3}$$

with the following properties

(i) $\tilde{x} = x^{1/m}$, $\tilde{\varepsilon} = \varepsilon^{1/\ell}$; m, ℓ are positive integers. H is a non-negative integer.

(ii) $\hat{Y}(\tilde{x}, \tilde{\varepsilon})$ and $Q(\tilde{x}, \tilde{\varepsilon})$ are matrices in class $\mathcal{A}^{**}$ in $|\arg x| \le \delta_1 \le \delta_0$.

(iii) $Q(\tilde{x}, \tilde{\varepsilon})$ is a polynomial in $\tilde{\varepsilon}$.

(iv) $Q(\tilde{x}, \tilde{\varepsilon})$ is diagonal.

(v) The determinant of the leading term $\hat{Y}_0(\tilde{x})$ in the expansion of $\hat{Y}(\tilde{x}, \tilde{\varepsilon})$ is not identically zero.

The method of proof is completely constructional. This does not exclude that for many concrete examples shorter procedures may be available. Most of the required operations are rational. There occur, in addition, calculations of eigenvalues of holomorphic matrix functions and quadratures of analytic functions.

The by now fairly common notation $A =: B$ will occasionally be used to indicate that the formula represents a *definition of* B. The meaning of $A := B$ is analogous.

§ 2. Asymptotic Expansions in Growing Domains.

The sequence of transformations of the differential equation (1.1) to be performed later in this paper, while simplifying it in essential respects, also complicates it by introducing divergent asymptotic series and negative powers of x. To cope with these difficulties, a theory of asymptotic series of the form $\sum_{r=0}^{\infty} A_r(x)\varepsilon^r$ will be developed in which the analytic functions $A_r(x)$ are allowed to have poles at $x = 0$. This theory can easily be extended to slightly more involved types of singularities. See section 9.

Definition 2.1. A matrix valued function $A(x, \varepsilon)$ will be said to belong to the class $\mathcal{O}$, if it has the following properties:

(i) A is holomorphic in both variables for

(2.1) $$0 < |x| \le x_0, \quad 0 < \varepsilon \le \varepsilon_0 ;$$

(ii) A has an asymptotic expansion, in a sense to be explained presently, of the form

(2.2) $$A(x, \varepsilon) \sim \sum_{r=0}^{\infty} A_r(x)\varepsilon^r, \quad \text{as} \quad \varepsilon \to 0+ ,$$

where

(2.3) $$A_r(x) = x^{-\sigma_r} \check{A}_r(x), \quad \check{A}_r(0) \ne 0, \quad r = 0, 1, \dots ,$$

whenever $A_r(x)$ is not identically zero. Here σ_r is an integer (not necessarily non-negative) and $\check{A}_r(x)$ is holomorphic in $|x| \le x_0$.

(iii) *There is a constant* κ, $0 < \kappa \leq \infty$ *such that*

(2.4) $$\sigma_r \leq r/\kappa, \quad r = 0, 1, \ldots .$$

(iv) *Define the domain* $\mathcal{S}_\varepsilon$ *by*

(2.5) $$\mathcal{S}_\varepsilon = \{x \mid t_0^{-1}\varepsilon^\kappa \leq |x| \leq x_0, \ |\arg x - \theta_0| \leq \delta_0\},$$

where $t_0 > 0$, $x_0 > 0$, $\delta_0 > 0$, *and* θ_0 *are certain constants. Then there is for every* $N \geq 0$ *a constant* c_N, *depending on* N, *but not on* x *or* ε, *such that*

(2.6) $$|A(x,\varepsilon) - \sum_{r=0}^{N} A_r(x)\varepsilon^r| \leq c_N(|x|^{-1/\kappa}\varepsilon)^{N+1},$$

for $x \in \mathcal{S}_\varepsilon$, $0 < \varepsilon \leq \varepsilon_0$.

Remarks:

(a) The inequality (2.4) means that the order of the poles of $A_r(x)$ can grow, at worst, linearly with r. This restraining inequality is the stronger the larger κ is. The least upper bound of the possible values for the number κ will therefore be called the restraint index of the series.

(b) The constant θ_0 will be held fixed throughout the arguments to come. It can be made equal to zero by a rotation of the x-plane. Therefore, the condition

(2.7) $$\theta_0 = 0$$

does not restrict the generality of the paper. It will be imposed from now on. If it is desirable to specify the other constants, the corresponding subfamily of $\mathcal{A}$ will be designated by $\mathcal{A}(t_0, x_0, \varepsilon_0, \delta_0, \kappa)$. Decreasing

any of these constants narrows the family. For simplicity, it will be assumed that $x_0 < 1$, although this is not essential. Then one must also have $t_0^{-1}\varepsilon_0^{\kappa} < x_0 < 1$.

(c) In subdomains of $\mathscr{D}_\varepsilon$ where $|x| \geq x_1$, x_1 independent of ε, the error in (2.6) is uniformly $O(\varepsilon^{N+1})$, as for ordinary asymptotic series. At the other extreme, the best that can be said uniformly in the whole domain $\mathscr{D}_\varepsilon$ is that the error is $O(t_0^{\kappa(N+1)})$, as $t_0 \to 0$. In the subdomain of $\mathscr{D}_\varepsilon$ defined by $|x| \geq t_0^{-1}\varepsilon^{\kappa'}$, with $\kappa' < \kappa$, the right member of (2.6) becomes $O(\varepsilon^{(1-\kappa'/\kappa)(N+1)})$, as $\varepsilon \to 0+$.

(d) If $A(x,\varepsilon)$ is in $\mathscr{A}$, the matrix $x^{-k}A(x,\varepsilon)$, with $k > 0$, is not, unless $A_0(x) \equiv 0$. However, $\varepsilon^h x^{-k} A(x,\varepsilon)$, with $h > 0$, belongs to $\mathscr{A}$ with a restraint index $\kappa' \geqq \min(\kappa, h/k)$. This is most easily verified by a geometric interpretation of (2.4): The points $(0,0)$, $(1,\sigma_1)$, $(2,\sigma_2)$, etc. in the (r,σ)-plane lie on or below the line $r - \kappa\sigma = 0$ and in the right half-plane. Multiplication by $\varepsilon^h x^{-k}$ translates this point set by the vector (h,k).

The lemma below is almost obvious, and the proof will be omitted.

Lemma 2.1. *If A and B are in $\mathscr{A}$, then $A+B$ and AB are in $\mathscr{A}(t_0, x_0, \varepsilon_0, \delta_0, \kappa)$ with values of the constants that are the smaller one of the corresponding values for A and B, respectively.*

Lemma 2.2. *If $A \in \mathscr{A}(t_0, x_0, \varepsilon_0, \delta_0, \kappa)$ and $A_0(0)$ is invertible, then $A^{-1} \in \mathscr{A}(\tilde{t}_0, \tilde{x}_0, \tilde{\varepsilon}_0, \delta_0, \kappa)$ with constants $\tilde{t}_0, \tilde{x}_0, \tilde{\varepsilon}_0$* that do not exceed t_0, x_0, ε_0.

Proof. By continuity, $A_0(x)$ is invertible for small x. If $\tilde{t}_0$ is small enough, the relation (2.6) for $N = 0$ implies then that $A(x, \varepsilon)$ is invertible (see the Remark (c), above). To calculate an asymptotic series for $A^{-1}(x, \varepsilon)$, one first determines recursively a sequence of matrices $B_r(x)$, $r = 0, 1, \ldots$, from the formal relation

$$\left(\sum_{r=0}^{\infty} A_r(x)\varepsilon^r\right)\left(\sum_{r=0}^{\infty} B_r(x)\varepsilon^r\right) = I\,,$$

whose expansion leads to the relations

$$B_0 = A_0^{-1};\ B_r = A_0^{-1} \sum_{j=0}^{r-1} A_{r-j} B_j, \qquad r = 1, 2, \ldots\,.$$

They imply, by means of a simple induction, that

$$(2.8) \qquad B_r(x) = x^{-\sigma r} \check{B}_r(x),\ \check{B}_r(x) \text{ holomorphic in } |x| \le \tilde{x}_0\,.$$

Now define the remainders $R_N(x, \varepsilon)$, $S_N(x, \varepsilon)$ by

$$R_N(x, \varepsilon) = A(x, \varepsilon) - \sum_{r=0}^{N} A_r(x)\varepsilon^r$$

$$S_N(x, \varepsilon) = A^{-1}(x, \varepsilon) - \sum_{r=0}^{N} B_r(x)\varepsilon^r\,.$$

Then the identity $AA^{-1} = I$ can be written as

$$AS_N = I - \left(\sum_0^N A_r\varepsilon^r\right)\left(\sum_0^N B_r\varepsilon^r\right) - R_N \sum_0^N B_r\varepsilon^r\,,$$

or, in consequence of the definition of the B_r, as

$$(2.9) \qquad S_N = A^{-1} \sum_{\substack{r+s>N \\ r,s \le N}} A_r B_s \varepsilon^{r+s} - A^{-1} R_N \sum_0^N B_r \varepsilon^r\,.$$

The summation in the right member satisfies the inequality

$$\Big| \sum_{\substack{r+s>N \\ r,s\le N}} A_r(x)B_s(x)\varepsilon^{r+s} \Big| \le K_N |x|^{-r/\kappa - s/\kappa}\varepsilon^{r+s} \le K_N[\,|x|^{-1/\kappa}\varepsilon\,]^{N+1}$$

with some constant K_N, by virtue of (2.3), (2.4) and (2.8). Now, $A^{-1}(x,\varepsilon)$ is bounded in $\mathcal{S}_\varepsilon$, for $0 < \varepsilon \le \varepsilon_0$, and so is $\sum_{r=0}^{N} B_r(x)\varepsilon^r$. Also, $R_N(x,\varepsilon) = 0((|x|^{-1/\kappa}\varepsilon)^{N+1})$ in that domain. Hence, (2.9) establishes the same order of magnitude for $S_N(x,\varepsilon)$, which proves the Lemma.

Lemma 2.3. If $A(x,\varepsilon) \in \mathcal{Q}(t_0, x_0, \varepsilon_0, \delta_0, \kappa)$ and $\tilde{t}_0$, $\tilde{x}_0$, $\tilde{\delta}_0$ are any positive constants less than t_0, x_0, δ_0, then $dA(x,\varepsilon)/dx \in \mathcal{Q}(\tilde{t}_0, \tilde{x}_0, \varepsilon_0, \delta_0, \frac{\kappa}{\kappa+1})$ and $xdA(x,\varepsilon)/dx \in \mathcal{Q}(\tilde{t}_0, \tilde{x}_0, \varepsilon_0, \tilde{\delta}_0, \kappa)$. The asymptotic series for the derivative is the termwise derivative of the series for $A(x,\varepsilon)$. (If $x = \infty$, one may replace $\kappa/(\kappa+1)$ with ∞.

Proof. One has

$$(2.10)\qquad dA(x,\varepsilon)/dx = dA_0(x)/dx + \sum_{r=1}^{N} x^{-(\sigma_r+1)}\hat{A}_r(x)\varepsilon^r + dR_N(x,\varepsilon)/dx\,,$$

with

$$\hat{A}_r(x) = -\sigma_r\check{A}_r(x) + xd\check{A}_r(x)/dx\,.$$

(The summation in (2.10) is absent for $N = 0$.) Let $\tilde{\mathcal{S}}_\varepsilon$ be the subdomain of $\mathcal{S}_\varepsilon$ defined by

$$\mathcal{S}_\varepsilon = \{x \,|\, \tilde{t}_0^{-1}\varepsilon^\kappa \le |x| \le \tilde{x}_0,\ |\arg x| \le \tilde{\delta}_0\}\,,$$

where

$$\tilde{t}_0 = t_0(1-\gamma),\ \tilde{x}_0 = x_0(1+\gamma)^{-1},\ \tilde{\delta}_0 = \delta_0 - \arcsin\gamma ,$$

and γ is an arbitrarily small positive constant. Around a given $x \in \tilde{\mathcal{S}}_\varepsilon$ as center describe a closed disk T_x of radius $|x|\gamma$. Then, for every $\xi \in T_x$ one has

$$t_0^{-1}\varepsilon^\kappa = \tilde{t}_0^{-1}\varepsilon^\kappa(1-\gamma) \leq |x|(1-\gamma) \leq |\xi| \leq |x|(1+\gamma) \leq \tilde{x}_0(1+\gamma) \leq x_0$$

and

$$|\arg\xi - \arg x| \leq \arcsin\gamma ,$$

so that $T_x \in \mathcal{S}_\varepsilon$, for $x \in \tilde{\mathcal{S}}_\varepsilon$. Hence, by the Cauchy inequality and (2.6),

$$(2.11)\quad |dR_N(x,\varepsilon)/dx| \leq \max_{\xi\in T_x} |R_N(\xi,\varepsilon)|/|x|\gamma \leq \tilde{C}_N|x|^{-(N+1)/\kappa-1}\varepsilon^{N+1} .$$

The constant $\tilde{C}_N$ depends on γ.

From (2.4) it follows that $\sigma_r + 1 \leq r/\kappa + 1 \leq r/\frac{\kappa r}{\kappa+r} \leq r/\frac{\kappa r}{\kappa+1}$, for $r > 0$. This shows that the termwise differentiated series for $A(x,\varepsilon)$ satisfies (2.4) with κ replaced by $\kappa/(\kappa+1)$. Also, (2.11) implies that the analog of (2.6) for $dA(x,\varepsilon)/dx$ is true with the same substitution for κ. Thus, the statement of the lemma for dA/dx itself is proved. The statement for $x\,dA(x,\varepsilon)/dx$ is proved by going through the same steps after multiplication with x.

Lemma 2.4. *Let*

$$A_r(x) = x^{-\sigma_r}\check{A}_r(x),\ r \geq 0 ,$$

where the σ_r are integers such that

$$\sigma_r \leq r/\kappa$$

for some restraint index κ *in* $0 < \kappa \leq \infty$ *and that the* $\check{A}_r(x)$ *are holomorphic in* $|x| \leq x_0 < 1$, *with* $\check{A}_r(0) \neq 0$, *unless* $A_r(x) \equiv 0$. *Then there exists in every family* $\mathcal{A}(t_0, x_0, \delta_0, \varepsilon_0, \kappa)$ *with* $t_0 < 1$ *a function* $A(x, \varepsilon)$ *with the asymptotic expansion* $\sum_{r=0}^{\infty} A_r(x)\varepsilon^r$.

Proof. This lemma is a generalization of the Borel-Ritt Theorem on the existence of asymptotic series (see [15] or Thm. 9.6 in [18]). Let

$$|A_r(x)| \leq a_r \quad \text{for} \quad |x| \leq x_0 , \tag{2.12}$$

and define $\alpha_r(x, \varepsilon)$ by

$$\alpha_r(x, \varepsilon) = 1 - \exp\{-a_r^{-1}(x^{-1/\kappa}\varepsilon)^{-\beta}\} , \tag{2.13}$$

whenever $a_r \neq 0$. The positive constant β is to be chosen so small that the exponent in braces is in the left half plane for $|\arg x| \leq \delta_0$. Then

$$|\alpha(x, \varepsilon)| \leq a_r^{-1}(|x|^{-1/\kappa}\varepsilon)^{-\beta} , \tag{2.14}$$

for $|\arg x| \leq \delta_0$. In the part of that sector where

$$t_0^{-1}\varepsilon^{\kappa} \leq |x| \leq x_0 \tag{2.15}$$

one has

$$|A_r(x)\alpha_r(x, \varepsilon)\varepsilon^r| \leq (|x|^{-1})^{\sigma r^{-\beta/\kappa}} \varepsilon^{r-\beta} \leq (|x|^{-1})^{(r-\beta)/\kappa}\varepsilon^{r-\beta} \leq t_0^{(r-\beta)/\kappa} .$$

Therefore, there is a function $A(x, \varepsilon)$ defined by the convergent series

$$A(x, \varepsilon) = \sum_{r=0}^{\infty} A_r(x)\alpha_r(x, \varepsilon)\varepsilon^r . \tag{2.16}$$

$A(x,\varepsilon)$ is holomorphic in $\mathfrak{R}_\varepsilon$ for $0 < \varepsilon \le \varepsilon_0$. (Observe, again, that one must choose ε_0 so small that $t_0^{-1}\varepsilon_0^\kappa < x_0 < 1$).

To prove that $A(x,\varepsilon) \sim \sum_{r=0}^{\infty} A_r(x)\varepsilon^r$, in the sense of the definition (2.6), one proceeds as follows.

$$
(2.17)\qquad
\begin{aligned}
\Big|A(x,\varepsilon) - \sum_{r=0}^{N} A_r(x)\alpha_r(x,\varepsilon)\varepsilon^r\Big| &\le \sum_{r=0}^{N} a_r|x|^{-\sigma_r}|\exp\{-a_r^{-1}(x^{-1/\kappa}\varepsilon)^{-\beta}\}|\varepsilon^r \\
&+ \sum_{r=N+1}^{\infty} a_r|x|^{-\sigma_r}|\alpha_r(x,\varepsilon)|\varepsilon^r =: \Sigma_1 + \Sigma_2 .
\end{aligned}
$$

For abbreviation, set

$$\zeta = x^{-1/\kappa}\varepsilon .$$

Then

$$(2.18)\qquad x_0^{-1/\kappa}\varepsilon \le |\zeta| \le t_0^{1/\kappa},$$

for x in $t_0^{-1}\varepsilon^\kappa \le |x| \le x_0$, and also

$$|x|^{-\sigma_r}\varepsilon^r \le |\zeta|^r .$$

Hence,

$$(2.19)\qquad \Sigma_1 \leqq |\zeta|^{N+1}\sum_{r=0}^{N} a_r\exp\{-a_r^{-1}\mathrm{Re}(\zeta^{-\beta})\}|\zeta|^{r-N-1} \leqq |\zeta|^{N+1}C_{1N},$$

for $|\zeta|$ in the interval (2.18). The constant C_{1N} is independent of ζ. Finally,

$$
(2.20)\qquad
\begin{aligned}
\Sigma_2 &\leqq 2a_{N+1}|\zeta|^{N+1} + |\zeta|^{N+1}\sum_{r=N+2}^{\infty}|\zeta|^{r-N-1-\beta} \\
&= |\zeta|^{N+1}\Big(2a_{N+1} + \frac{|\zeta|^{1-\beta}}{1-|\zeta|}\Big) \le C_{2N}|\zeta|^{N+1} .
\end{aligned}
$$

The proof of the lemma is completed by inserting (2.19) and (2.20) into (2.17).

§ 3. The Program.

The more general types of differential equations to which one is led in the reduction process to be performed on the original equation (1.1) have the form

$$\varepsilon^h x^{-k} \frac{dy}{dx} = A(x, \varepsilon) y , \tag{3.1}$$

where h and k are integers, $h \geq 0$, and $A(x, \varepsilon)$ is of class $\mathcal{A}$. For equation (1.1),

$$h > 0, \ k = 0, \ \kappa = \infty . \tag{3.2}$$

Incidentally, with the substitutions $x = t\varepsilon^{\kappa}$ and

$$A_r(x)\varepsilon^r = \check{A}_r(x) t^{-\sigma_r} \varepsilon^{r-\sigma_r \kappa}$$

in the series (2.2) for $A(x, \varepsilon)$, the differential equation (3.1) can be interpreted as involving two independent variables a "fast" one, t, and a "slow" one, x. This is an approach used in much of the literature, e.g., in [19].

The sequence of transformations through which one reaches, eventually, a form amenable to elementary solution may be long. To avoid an awkward proliferation of the notation, the previous notation will be re-adopted after each transformation. The simplification of the problem reached at each stage appears then as a property that may be assumed without loss of generality.

It will be helpful to begin with a list of the types of transformations to be performed, to number them and to give them descriptive names.

Type I (Transformation by power series): Set

$$y = P(x, \varepsilon)\tilde{y} ,$$

with $P \in \mathcal{A}$ and $P(0, \varepsilon)$ invertible for small $\varepsilon > 0$.

Type II (Parameter shearing):

$$y = \operatorname{diag}(\varepsilon^{\alpha_1}, \varepsilon^{\alpha_2}, \ldots, \varepsilon^{\alpha_n})\tilde{y}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are certain rational numbers.

Type III (Independent variable shearing):

$$y = \operatorname{diag}(x^{\beta_1}, x^{\beta_2}, \ldots, x^{\beta_n})\tilde{y}$$

with certain rational numbers $\beta_1, \beta_2, \ldots, \beta_n$.

Type IV (Re-adjustment of the parameter):

$$\varepsilon = \tilde{\varepsilon}^m, \quad m \text{ a positive integer} .$$

Type V (Re-adjustment of the independent variable):

$$x = \tilde{x}^m, \quad m \text{ a positive integer.}$$

Type VI (Eigenvalue shifting):

$$y = \exp\{\varepsilon^{-h} \int \lambda(x) x^k dx\}\tilde{y} ,$$

where $\lambda(x)$ is a holomorphic scalar function (usually a constant).

A transformation of type I changes (3.1) into

$$\varepsilon^h x^{-k} \frac{d\tilde{y}}{dx} = (P^{-1}AP - \varepsilon^h x^{-k} P^{-1} \frac{dP}{dx})\tilde{y} . \tag{3.3}$$

Lemma 3.1. If A and P are in $\mathcal{A}$ with restraint indices κ_A, κ_P, if $P(0, \varepsilon)$ is invertible for small positive ε, and if $h > 0$, then

$P^{-1}AP - \varepsilon^h x^{-k} P^{-1} \frac{dP}{dx}$ <u>is in</u> $\mathcal{O}$ <u>with a restraint index</u> κ <u>that satisfies the inequalities</u>

$$\kappa \geq \min(\kappa_A, \kappa_P, \frac{h}{k+1})$$

(3.4) and

$$\kappa \geq \min(\kappa_A, \kappa_P/(\kappa_P + 1), \frac{h}{k})$$

when $k \geq 0$. If $\kappa_P = \infty$, the middle term in both inequalities may be omitted. If $k \leq 0$ the last term can be omitted in (3.4).

Proof. Write $\varepsilon^h x^{-k} \frac{dP}{dx} = \varepsilon^h x^{-k-1}(x \frac{dP}{dx})$, then the lemma follows immediately from Lemmas 2.1, 2.2, 2.3 and Remark (d) after Definition 2.1.

§4. An Approximate Block-diagonalization.

In [16] Sibuya proved a fundamental block-diagonalization theorem for differential equations of the form (1.1). In this and the following section Sibuya's result is being extended to the more general differential equation (3.1).

The block diagonalization is indicated only when

$$h > 0 \tag{4.1}$$

and if $A_0(0)$ in (2.2) has at least two distinct eigenvalues. What follows applies only in this case.

It is clear that a suitable transformation of Type I with constant coefficients will change $A_0(0)$ into its Jordan canonical form. This transformation does not affect the restraint index. According to the program explained in section 3 it may be stipulated that $A_0(0)$ has already the property below.

Property 1: $A_0(0)$ is in Jordan canonical form.

Since $A_0(0)$ has more than one eigenvalue it can be written - not uniquely - in the partitioned form

$$A_0(0) = \begin{bmatrix} A_0^{11}(0) & 0 \\ 0 & A_0^{22}(0) \end{bmatrix} \tag{4.2}$$

with $A_0^{11}(0)$ $A_0^{22}(0)$ being Jordan matrices without a common eigenvalue.

There is an often proved theorem to the effect that there exists a holomorphic nonsingular matrix $P(x)$ in $|x| \leq \tilde{x}_0$ $(0 < \tilde{x}_0 \leq x_0)$ such

that $P^{-1}(x)A_0(x)P(x)$ is block-diagonal in the same partitioning as that in (4.2). (See, e.g., [4]). A transformation of Type I with this matrix P replaces $A_0(x)$ by a block diagonal matrix, as described. Thus, one can stipulate

Property 2:

$$A_0(x) = \begin{bmatrix} A_0^{11}(x) & 0 \\ 0 & A_0^{22}(x) \end{bmatrix},$$

where $A_0^{jj}(0)$, $j = 1, 2$, are in Jordan form and have distinct eigenvalues.

Note that this transformation may have lowered the restraint index to h/k, if $k > 0$. (See Lemma 3.1).

Now, a matrix $P(x, \varepsilon) \in \mathcal{A}$ will be constructed such that the new coefficient matrix

$$B = P^{-1}AP - \varepsilon^h x^{-k} P^{-1} \frac{dP}{dx} \tag{4.3}$$

of the differential equation

$$\varepsilon^h x^{-k} \frac{dz}{dx} = B(x, \varepsilon) z \tag{4.4}$$

resulting from (3.1) by the transformation

$$y = P(x, \varepsilon) z \tag{4.5}$$

has an expansion

$$B(x, \varepsilon) \sim \sum_{r=0}^{\infty} B_r(x) \varepsilon^r$$

which has all its coefficients $B_r(x)$ block diagonal in the partition

defined by (4.2). To that end, one inserts the series

(4.7) $$P(x, \varepsilon) \sim \sum_{r=0}^{\infty} P_r(x)\varepsilon^r ,$$

as well as the series in (4.6) into (4.3) re-written in the form

(4.8) $$\varepsilon^h x^{-k} \frac{dP}{dx} = AP - PB$$

and identifies like powers of ε in the two members.

The first of the recursive sequence of conditions so obtained is

$$A_0 P_0 - P_0 B_0 = 0 .$$

It is satisfied by setting

(4.9) $$P_0 = I, \ B_0 = A_0 .$$

The subsequent conditions then become

(4.10) $$A_0 P_r - P_r A_0 = B_r + H_r, \quad r > 0$$

with

(4.11) $$H_r = \sum_{\substack{\nu + \mu = r \\ \mu, \nu < r}} (P_\mu B_\nu - A_\nu P_\mu) - A_r - x^{-k} \frac{dP_{r-k}}{dx} .$$

The last term in (4.11) is absent for $r \leq h$.

The successive calculation of the matrices B_r, $r = 1, 2, \ldots,$ from (4.10), (4.11) so that they are all block diagonal follows almost exactly Sibuya's procedure in [16] as presented in [18], §§11 and 26. One desires to have

(4.12) $$B_r = \begin{bmatrix} B_r^{11} & 0 \\ 0 & B_r^{22} \end{bmatrix}, \quad r > 0 .$$

This can, indeed, be achieved, even by means of matrices P_r of the special form

(4.13)
$$P_r = \begin{bmatrix} 0 & P_r^{12} \\ P_r^{21} & 0 \end{bmatrix},$$

as can be seen by inserting (4.12) and (4.13) into (4.10). One finds the four conditions (cf. [18], formula (11.15))

(4.14)
$$\begin{aligned} 0 &= B_r^{11} + H_r^{11} \qquad & A_0^{11}P_r^{12} - P_r^{12}A_0^{22} &= H_r^{12} \\ A_0^{22}P_r^{21} - P_r^{21}A_0^{11} &= H_r^{21} \qquad & 0 &= B_r^{22} + H_r^{22} \end{aligned}.$$

If B_j, P_j have already been calculated for $j < r$, the matrix H_r is known. Since A_0^{11}, A_0^{22} have no common eigenvalues for small x, the matrices P_r^{12} and P_r^{21} can be calculated uniquely from the off-diagonal relations in (4.14). The matrices B_r^{11}, B_r^{22} are given directly by the diagonal equations in (4.14).

The matrices P_r, B_r so determined have poles at $x = 0$ and it must still be shown that the orders of these poles grow at most linearly with r, i.e., that (2.4) is true for some restraint index κ.

It is clear from (4.14) that if H_r is already known to have a pole of order σ_r, then P_r and B_r have poles of at most order σ_r. By induction, it follows from (4.11) and (4.14) that the inequality (2.4) is true, with the restraint index κ of A, up to $r = h$. Beyond that one has the

<u>Lemma 4.1</u>. <u>The series</u> (4.6) <u>and</u> (4.7) <u>constructed by solving</u> (4.9), (4.14)

satisfy (2.4) with a restraint index $\kappa = \kappa_P$ which satisfies the inequality

$$\kappa_P \geq \min(\kappa_A, \frac{h}{k+1}), \quad \text{when} \quad k \geq 0 \tag{4.15}$$

and $$\kappa_P \geq \kappa_A, \quad \text{when} \quad k < 0,$$

where κ_A is the restraint index of A.

Proof. Assume $k \geq 0$. The assertion is already known to be true for $r \leq h$. Assume it to be true for all subscripts below a certain r. Then H_r in (4.11) can have a pole whose order is at most

$$\max_{\substack{\nu+\mu=r \\ \nu,\mu<r}} \left[\frac{\mu}{\kappa'} + \frac{\nu}{\kappa'}, \frac{r}{\kappa_A}, k+1+\frac{r-h}{\kappa'}\right], \tag{4.16}$$

where κ' is the right member of (4.15). If $\kappa' = \frac{h}{k+1}$, then $\kappa' < \kappa_A$ and $k+1+\frac{r-h}{\kappa'} = \frac{r}{h}(k+1) = \frac{r}{\kappa}$. Hence the number in (4.16) equals r/κ'. If $\kappa' = \kappa_A$, one has $\kappa' \leq \frac{h}{k+1}$, i.e. $k+1 \leq \frac{h}{\kappa'}$, so that $k+1+\frac{r-h}{\kappa'} \leq \frac{r}{\kappa'}$. Again (4.19) has the value $\frac{r}{\kappa'}$. As P_r and B_r do not have poles of higher order than has H_r, the lemma is proved for $k \geq 0$. For $k < 0$ one has, always, $\kappa' = \kappa_A$.

A reference to Lemma 2.4 now guarantees the existence of matrix functions $P(x,\varepsilon)$, $B(x,\varepsilon)$ in $\mathcal{A}$, with restraint index κ_P as in (4.15), such that $P_r(x)$ and $B_r(x)$ are the functions in (4.12) and (4.13), respectively. It does not follow, however, that these matrices P and B satisfy (4.3) exactly. It is true that the two members of (4.3) will have the same asymptotic expansion, but they may still differ by a function about which nothing is known except that its asymptotic series has all its terms zero.

According to the program of this paper, the result of this section will be formulated in the original notation as

Property 3.

$$A_r(x) = \begin{bmatrix} A_r^{11}(x) & 0 \\ 0 & A_r^{22}(x) \end{bmatrix}, \quad r = 0, 1, \ldots . \tag{4.17}$$

§ 5. Complete Blockdiagonalization.

Large parts of this section resemble the material in §§ 14 and 26 of [18], which may be consulted for a more detailed description of the method. See also [10] and [6].

Thanks to Lemma 2.4 and Property 3, above, there exists a truly blockdiagonal matrix $\tilde{A}(x, \varepsilon)$ in $\mathcal{O}$ such that

$$\tilde{A}^{jj}(x,\varepsilon) \sim \sum_{r=0}^{\infty} A_r^{jj}(x)\varepsilon^r, \qquad j = 1, 2 \tag{5.1}$$

in the sense of the definition in (2.6). Property 3 does not imply that $A(x, \varepsilon)$ is strictly blockdiagonal, but if

$$\tilde{A}(x,\varepsilon) := \begin{bmatrix} \tilde{A}^{11}(x,\varepsilon) & 0 \\ 0 & \tilde{A}^{22}(x,\varepsilon) \end{bmatrix} \tag{5.2}$$

then

$$A(x,\varepsilon) - \tilde{A}(x,\varepsilon) =: E(x,\varepsilon)\ , \tag{5.3}$$

with

$$E(x,\varepsilon) \sim 0\ . \tag{5.4}$$

The purpose of this section is to transform the differential equation (3.1) - whose coefficient matrix may now be assumed to satisfy (5.3) and (5.4) - into one with the strictly blockdiagonal matrix $\tilde{A}(x, \varepsilon)$. This is to be done by a transformation of Type I. In other words, a matrix $P \in \mathcal{O}$ must be found such that the coefficient matrix in (3.3) is equal to $\tilde{A}$ or, equivalently, that

$$\varepsilon^h x^{-k} P' = AP - P\tilde{A}\ . \tag{5.5}$$

In view of (5.3), (5.4) it is obvious that (5.5) can be formally satisfied by a series of the form given in (4.7) with $P_0(x) = I$, $P_r(x) = 0$, $r > 0$. It must be shown that among the infinitely many functions in $\mathcal{A}$ with that trivial asymptotic expansion there is one that satisfies (5.5) exactly. This will be done by following essentially Sibuya's procedure in [16]. (See also [18], §§12, 16).

Insert

$$P = I + \begin{bmatrix} 0 & \hat{P}^{12} \\ \hat{P}^{21} & 0 \end{bmatrix} \tag{5.6}$$

into (5.5). By an easy elimination there results then the differential equation of Riccati type,

$$\varepsilon^h x^{-k} \frac{d\hat{P}^{21}}{dx} = A^{22}\hat{P}^{21} - \hat{P}^{21}A^{11} + A^{21} - \hat{P}^{21}A^{12}\hat{P}^{21}, \tag{5.7}$$

for $\hat{P}^{21}$. An analogous equation holds for $\hat{P}^{12}$. The problem now consists in proving that (5.7) possesses a solution which is asymptotic to zero.

The notation will be simpler if the entries of the rectangular matrix $\hat{P}^{21}$ are arranged - no matter in what order - into an m-dimensional vector to be called w. Equation (5.7) can then be written as

$$\varepsilon^h x^{-k} \frac{dw}{dx} = f(x, w, \varepsilon) \tag{5.8}$$

with a right member that has the following properties:

(i) f is quadratic in w, in the sense that

$$f(x, w, \varepsilon) = a(x, \varepsilon) + M(x, \varepsilon)w + g(x, w, \varepsilon), \tag{5.9}$$

where M is an $m \times m$ matrix and the components of the vector g are quadratic forms in the components w_j, $j = 1, 2, \ldots, m$ of w.

(ii) The vector function $a(x, \varepsilon)$ is in $\mathcal{a}$ and is asymptotic to zero. The reason for the last assertion is that $A^{21}(x, \varepsilon) \sim 0$ in (5.7), by (5.2), (5.3) and (5.4).

(iii) $M(x, \varepsilon) \in \mathcal{a}$. Let

(5.10)
$$M(x, \varepsilon) \sim \sum_{r=0}^{\infty} M_r(x)\varepsilon^r$$

be the asymptotic expansion of $M(x, \varepsilon)$. Then $M_0(0)$ is non-singular. This reflects the fact that $A_0^{11}(0)$ and $A_0^{22}(0)$ have no common eigenvalues, according to Property 2 in section 4. No generality is lost by assuming $M_0(0)$ to be in Jordan form, since this requires only a preliminary linear transformation of w with a constant coefficient matrix.

(iv) Let

(5.11)
$$g_j(x, \varepsilon, w) := \sum_{\alpha, \beta=1}^{m} g_{j\alpha\beta}(x, \varepsilon) w_\alpha w_\beta, \quad j = 1, 2, \ldots, m$$

be the components of g in (5.9). Then $g_{j\alpha\beta} \in \mathcal{a}$, and $g_{j\alpha\beta} \sim 0$. Again, the last mentioned fact is a consequence of the fact that the off-diagonal blocks of $A(x, \varepsilon)$ are asymptotic to zero.

Following a standard procedure, the differential equation (5.8) is now written in the form

(5.12)
$$\varepsilon^h x^{-k} \frac{dw}{dx} = M_0(0)w + p(x, w, \varepsilon)$$

with

$$(5.13)\qquad p(x, w, \varepsilon) = a(x, \varepsilon) + (M(x, \varepsilon) - M_0(0))w + g(x, w, \varepsilon)$$

and then converted into the equivalent integral equation

$$(5.14)\quad w(x, \varepsilon) = \varepsilon^{-h} \int_{\Gamma(x)} \exp\{[q(x) - q(\zeta)] M_0(0)\varepsilon^{-h}\} p(\zeta, w(\zeta, \varepsilon), \varepsilon)\zeta^k d\zeta \,.$$

Here,

$$q(x) = x^{k+1}/(k+1), \quad \text{for} \quad k \neq -1, \quad \text{and} \quad q(x) = \log x, \quad \text{for} \quad k = -1 \,.$$

The symbol $\Gamma(x)$ stands for a set of m paths $\gamma_j(x)$, $j = 1, 2, \dots, m$, all ending at x, each to be used for the integration of the corresponding component of the integrand vector.

The construction of regions and paths suitable for the asymptotic analysis of the integral equation (5.13) requires some geometric preparations:

The mapping

$$\tilde{\zeta} = q(\zeta), \quad \text{with} \quad \tilde{x} = q(x)$$

takes the domain $\mathcal{D}_\varepsilon$ in the ζ-plane into a domain $q(\mathcal{D}_\varepsilon)$ in the $\tilde{\zeta}$-plane. For $k \geq 0$, the latter domain is again a sector of an annulus near the origin, as in (2.5). If $k = -1$, it is a rectangle in the left half plane and for $k < -1$ a sector of an annulus in the left half-plane, close to the point at infinity for small ε. If λ_j, $j = 1, 2, \dots, m$, are the eigenvalues of $M_0(0)$ - none of which are zero - there are finitely many "exceptional" directions in the $\tilde{\zeta}$-plane along which $\mathrm{Re}(\tilde{\zeta}\lambda_j)$ is constant for at least one j. Now construct a rhombus $\tilde{\mathcal{R}}$ in $q(\mathcal{D}_\varepsilon)$ according to the following rule. If the bisecting ray of $q(\mathcal{D}_\varepsilon)$ does not have exceptional direction, then $q(x_0)$ and $q(t_0^{-1}\varepsilon^\kappa)$ are two opposite vertices of $\tilde{\mathcal{R}}$ and the interior angles of $\tilde{\mathcal{R}}$ at these vertices must be so

small that no directions from these vertices into $\tilde{\mathcal{R}}$ are exceptional. If the bisecting ray of $q(\mathcal{S}_\varepsilon)$ __is__ exceptional, the construction must be modified by rotating the diagonal of $\tilde{\mathcal{R}}$ slightly with respect to the real $\tilde{\zeta}$-axis, but keeping the two opposite vertices on the boundary of $q(\mathcal{S}_\varepsilon)$.

Next, choose x_0^*, t_0^* and δ_0^* so small that the image $q(\mathcal{S}_\varepsilon^*)$ of the domain

$$\mathcal{S}_\varepsilon^* = (x \,|\, t_0^{*-1}\varepsilon^\kappa \le |\zeta| \le x_0^*, \ |\arg \zeta| \varepsilon \delta_0^*)$$

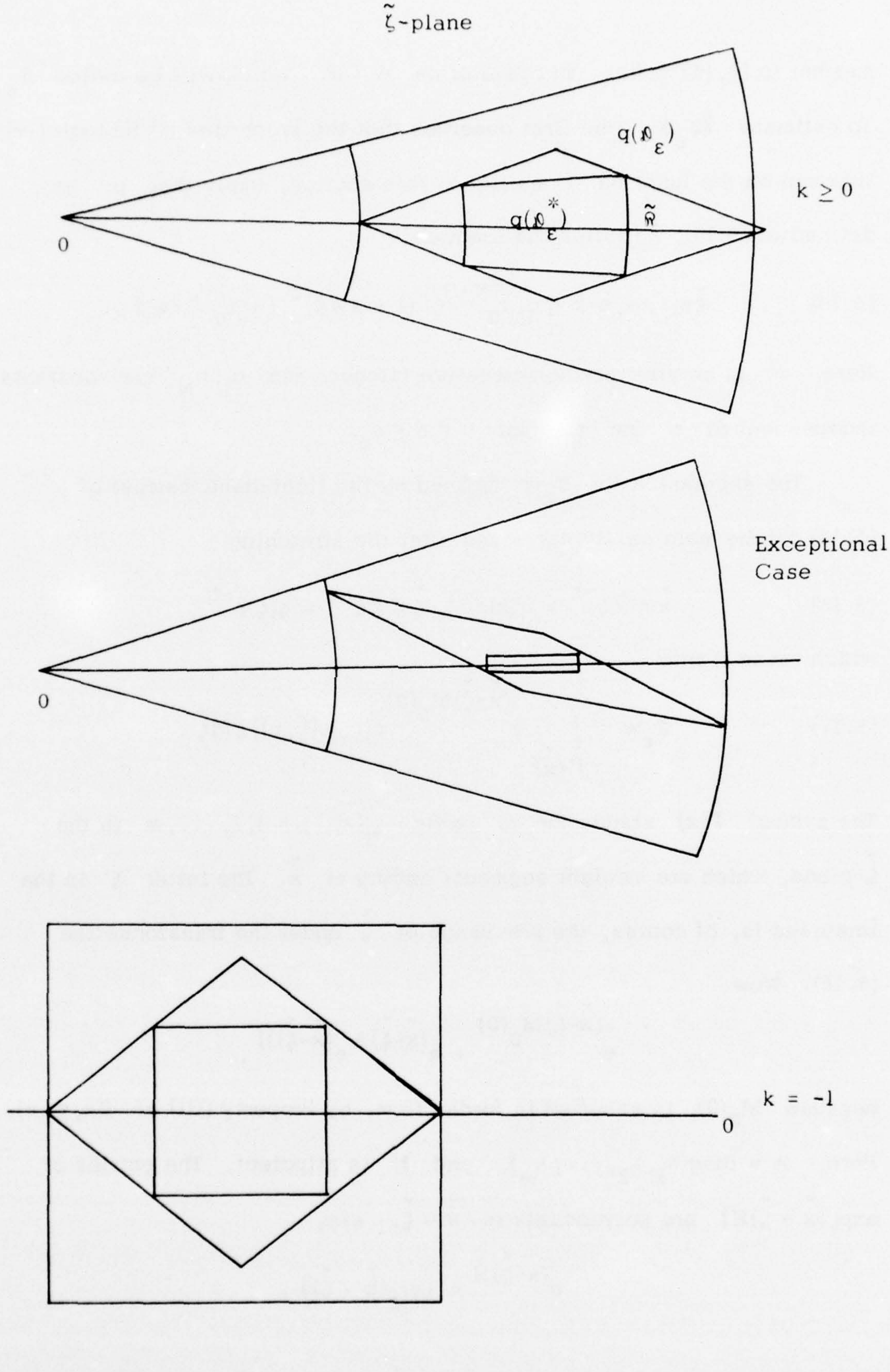
in the $\tilde{\zeta}$-plane lies inside $\tilde{\mathcal{R}}$. The pre-image of $\tilde{\mathcal{R}}$ will be denoted by $\mathcal{R}$. (See the figure, below. The case $k > -1$ has appearance similar to $k \ge 0$, except that the sector opens to the left.)

For the remainder of this proof, x is to be an arbitrary point of $\mathcal{R}$. If $\tilde{x}_1$, $\tilde{x}_2$ are the vertices of $\mathcal{R}$ and $\tilde{x} = q(x)$, then $\mathrm{Re}(\tilde{\zeta}\lambda_j)$ decreases strictly along one of the two directed segments $\overrightarrow{\tilde{x}_1\tilde{x}}$ or $\overrightarrow{\tilde{x}_2\tilde{x}}$ in the $\tilde{\zeta}$-plane. This was the purpose of the preceding construction. Let $\tilde{\gamma}_j(\tilde{x})$ be that segment. The path $\gamma_j(x)$ in the ζ-plane is its pre-image.

The existence of a solution of the integral equation (5.14) can be established by the contraction mapping theorem. Let $\mathcal{B}$ be the Banach space of vector valued functions v of x holomorphic in the closure $\bar{\mathcal{R}}$ of the rhombus $\mathcal{R}$ with the norm

$$\|v\| = \max_{x \in \bar{\mathcal{R}}} |v(x)|$$

where $|v(x)| =: \sum_{j=1}^{m} |v_j(x)|$. For each ε in $0 < \varepsilon \le \varepsilon_0$ the right



$\tilde{\zeta}$-plane
$q(\mathcal{D}_\varepsilon)$
$q(\mathcal{D}^*_\varepsilon)$
$\tilde{\mathcal{D}}$
$k \geq 0$
0
Exceptional Case
0
$k = -1$
0

member in (5.14) defines an operator on $w \in \mathcal{B}$, which will be called $\mathfrak{J}_\varepsilon$. To estimate $\|\mathfrak{J}_\varepsilon\|$, one first observes that the properties (i) through (iv) imposed on the function f earlier in this section, imply that p, as defined in (5.13), satisfies the inequality

$$\|p(\cdot, w, \varepsilon)\| \leq c_N t_0^{(N+1)/\kappa}(1 + \|w\|)^2 + c_1 t_0^{2/\kappa}\|w\| . \tag{5.15}$$

Here, N is an arbitrary non-negative integer, and c_1, c_N are constants independent of ε and t_0, for $0 < \varepsilon \leq \varepsilon_0$.

The expression for $\mathfrak{J}_\varepsilon w$ defined by the right-hand member of (5.14) can be more easily appraised after the stretching

$$\check{x} = \tilde{x}\varepsilon^{-h} = q(x)\varepsilon^{-h}, \quad \check{\zeta} = \tilde{\zeta}\varepsilon^{-h} = q(\zeta)\varepsilon^{-h} , \tag{5.16}$$

which takes it into

$$\mathfrak{J}_\varepsilon w = \int_{\check{\Gamma}(\check{x})} e^{(\check{x}-\check{\zeta})M_0(0)} p(\zeta, w(\zeta,\varepsilon), \varepsilon)d\check{\zeta} . \tag{5.17}$$

The symbol $\check{\Gamma}(x)$ stands for m paths $\check{\gamma}_j(\check{x})$, $j = 1, 2, \ldots, m$ in the $\check{\zeta}$-plane, which are straight segments ending at $\check{x}$. The letter ζ in the integrand is, of course, the pre-image of $\check{\zeta}$ under the transformation (5.16). Now

$$e^{(\check{x}-\check{\zeta})M_0(0)} = e^{(\check{x}-\check{\zeta})\Lambda} e^{(\check{x}-\check{\zeta})H} ,$$

because $M_0(0)$ is assumed in Jordan form, by Property (iii) of $f(x, w, \varepsilon)$. Here, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$, and H is nilpotent. The entries of $\exp[(\check{x} - \check{\zeta})H]$ are polynomials in $\check{x} - \check{\zeta}$, say,

$$e^{(\check{x}-\check{\zeta})H} = \{q_{j\ell}(\check{x} - \check{\zeta})\} .$$

Therefore, the j^{th} component of the vector $\mathfrak{T}_\varepsilon w$ in (5.17) is

$$(5.18) \qquad \int_{\check{\gamma}_j(\check{x})} e^{(\check{x}-\check{\zeta})\lambda_j} \sum_{\ell=1}^{m} q_{j\ell}(\check{x} - \check{\zeta}) p_\ell(\zeta, w(\zeta, \varepsilon), \varepsilon) d\check{\zeta} .$$

By construction, $\mathrm{Re}[(\check{x} - \check{\zeta})\lambda_j]$ is strictly negative, for all $x \in \bar{\mathfrak{R}}$, along the path of integration in (5.18). The rhombus $\mathfrak{R}$ depends on ε, but the eigenvalues λ_j, and therefore the directions of the paths $\check{\gamma}_j(\check{x})$, do not. It follows that there is a constant $\delta > 0$, independent of ε, t_0 and x_0, so that

$$(5.19) \qquad \sum_{j=1}^{m} |e^{(\check{x}-\check{\zeta})\lambda_j}| \sum_{\ell=1}^{m} |q_{j\ell}(\check{x} - \check{\zeta})| \le e^{-\delta|\check{x}-\check{\zeta}|}, \quad j = 1, \dots, m ,$$

for all $\check{x}$ in the image of $\bar{\mathfrak{R}}$ in the ζ-plane, and for all $\check{\zeta}$ on $\check{\gamma}_j(\check{x})$.

The estimates (5.15) and (5.19) suffice to appraise $\mathfrak{T}_\varepsilon w$ from (5.17). One finds

$$(5.20) \qquad \|\mathfrak{T}_\varepsilon w\| \le \|p(\cdot, w(\cdot, \varepsilon), \varepsilon)\| \int_0^\infty e^{-\delta\rho} d\rho$$

$$\le \delta^{-1}[c_N t_0^{(N+1)/\kappa}(1 + \|w\|^\ell) + c_1 t_0^{2/\kappa}\|w\|] .$$

The contraction property of $\mathfrak{T}_\varepsilon$ is proved in a similar way with the help of the inequality

$$(5.21) \quad \|g(x, w, \varepsilon) - g(x, v, \varepsilon)\| \le C_N t_0^{(N+1)/\kappa}[\max(\|w\|, \|v\|)] \|w - v\|) ,$$

which follows easily from the property (iv) of $f(x, w, \varepsilon)$. Formulas (5.21) and (5.13), together with (5.17), lead to

(5.22) $\|\mathfrak{J}_\varepsilon w - \mathfrak{J}_\varepsilon v\| \le \delta^{-1}[c_1 t_0^{2/\kappa} + C_N t_0^{(N+1)/\kappa} \max(\|w\|, \|v\|)] \|w - v\|)$

in a way that is analogous to how (5.20) was derived. The last inequality shows that $\mathfrak{J}_\varepsilon$ contracts the distances in any given ball, say in $\|w\| \le 1$, if t_0 is chosen sufficiently small. Also, one sees from (5.20) that

$$\|\mathfrak{J}_\varepsilon(0)\| \le \gamma < 1 ,$$

for small t_0. The contraction mapping theorem as stated, e.g., in [3], Ch. XII, now guarantees the existence of a unique holomorphic solution w of (5.12).

From the result thus obtained one concludes by a short calculation that

(5.23) $$\|w\| \le K_N t_0^N (t_0^N) \quad \text{for all } N, \quad \text{as } t_0 \to 0$$

for $x \in \mathcal{S}_\varepsilon$, $0 < \varepsilon < \varepsilon_0$, K_N indep. of ε. One needs only to replace $\|\mathfrak{J}_\varepsilon w\|$ in (5.20) by $\|w\|$ and to solve the quadratic inequality for $\|w\|$. This is, of course, not as strong as the claim that $w \sim 0$ in the sense of the definition (2.6), since $t_0 \ge |x|^{-1}\varepsilon^\kappa$ for $x \in \mathcal{S}_\varepsilon$ and $0 \le \varepsilon \le \varepsilon_0$.

However, if $0 < \kappa' < \kappa$ and x is restricted to the smaller sector $\mathcal{S}'_\varepsilon$ defined by

$$x \in \mathcal{S}'_\varepsilon = \{x/t_0^{-1}\varepsilon^{\kappa'} \le |x| \le x_0, \ |\arg x| \le \delta\} ,$$

then (5.23) does contain the desired information. In fact, $|x| \ge t_0^{-1}\varepsilon^{\kappa'}$ means that

$$|x| \ge (t_0 \varepsilon^{\kappa-\kappa'})^{-1} \varepsilon^\kappa .$$

Hence, (5.23) is valid with t_0 replaced by $t_0\varepsilon^{\kappa-\kappa'}$, so that

$$\|w\| \le K_M t_0^M (\varepsilon^{\kappa-\kappa'})^M, \quad K_M \text{ a constant .}$$

For any given integer $N \ge 0$ set $M > (N+1)/(\kappa - \kappa')$ and thus write the last inequality as

$$\|w\| \le K_N^* \varepsilon^{N+1} .$$

As x_0 was assumed to be less than one, $x_0^{-1/\kappa'} > 1$, so that, indeed, $\|w\| \le (K_N^* x_0^{-1/\kappa'})^{N+1}$ for all N, if $x \in \mathcal{D}'_\varepsilon$. The following theorem has now been proved.

Theorem 5.1. *If $A(x, \varepsilon)$ has Properties 1, 2, 3 as stated in section 4, there is a transformation of Type I which takes the differential equation (3.1) into a system with a blockdiagonal coefficient matrix*

$$\begin{bmatrix} A^{11}(x,\varepsilon) & 0 \\ 0 & A^{22}(x,\varepsilon) \end{bmatrix}, \quad A^{jj}(x,\varepsilon) \in \mathcal{A}, \quad j = 1, 2 . \tag{5.24}$$

By repeated application of Theorem 5.1 the given system of differential equations may be split into a set of uncoupled systems of lower order, each of which has a leading coefficient matrix with only one distinct eigenvalue at $x = 0$. This eigenvalue can be reduced to zero by a transformation of Type VI. Accordingly, only differential equations that have the property below need to be considered from here on, at least when (4.1) is true.

Property 4. *When $h > 0$, then $A_0(0)$ is nilpotent and in Jordan form.*

6. Simplification by Arnold's Theory.

In [1], V. I. Arnold has developed a similarity theory for holomorphic matrix functions of one or several variables, and in [22] I have shown that this theory is useful for the asymptotic simplification of differential equations of the form (1.1).

Here is a very brief description of Arnold's canonical form for the matrix $A_0(x)$, which is assumed to have Property 4 stated at the end of section 5.

Let

$$m_1 \geq m_2 \geq \cdots \geq m_p, \quad m_1 + m_2 + \cdots + m_p = n, \tag{6.1}$$

be the degrees of the elementary divisors of $A_0(0)$, and assume - this implies no loss of generality - that the corresponding diagonal blocks of $A_0(0)$ are arranged in this order of decreasing size. Denote by $A_0^{\mu\nu}(0)$, $\mu, \nu = 1, 2, \ldots, p$, the rectangular blocks in the partition of $A_0(0)$ corresponding to this Jordan form, and, more generally, denote by $M^{\mu\nu}$ the blocks in the analogous partition of any $n \times n$ matrix M.

Arnold's theorem, applied to the matrix $A_0(x)$ says that it is holomorphically similar, in a sufficiently small disk $|x| \leq x_0$, to a matrix $\tilde{A}_0(x)$ in which $\tilde{A}_0^{\mu\nu}(x)$ has non-zero entries in its last row only, when $\mu \leq \nu$, and in its first column only, when $\mu > \nu$. A matrix of this structure will be called an Arnold canonical matrix. It will again be assumed that this reduction of $A_0(x)$ has already been performed by a

transformation of Type I independent of ε. In other words, Property 4 of section 5 will be strengthened to

Property 5. $A_0(0)$ is nilpotent, and $A_0(x)$ is in Arnold's canonical form.

This result can be improved by the method of [22], as will now be explained. To that end it is useful to introduce certain constant $n \times n$ matrices Γ_j, $j = 1, 2, \ldots, d$. The Γ_j have only one non-zero entry, and that entry is equal to 1. It is located in the last row of the block $\Gamma_j^{\mu\nu}$ when $\mu \leq \nu$ and in the first column when $\mu < \nu$. All such matrices are contained in the set $\Gamma_1, \ldots, \Gamma_d$. The lemma below, worded somewhat differently, can be found in [1].

Lemma 6.1 (Arnold). Corresponding to every constant complex $n \times n$ matrix H there exists an $n \times n$ matrix X and d scalars ρ_j, $j = 1, 2, \ldots, d$ such that

$$A_0(0)X - XA_0(0) - \sum_{j=1}^{d} \rho_j \Gamma_j = H \,. \tag{6.2}$$

(X and the ρ_j are not unique.)

Property 5 and Lemma 6.1 form the basis of a further simplification of the coefficient matrix $A(x, \varepsilon)$, at least when $A_0(0)$ is not zero. (If $A_0(0) = 0$, the $p = n$, $d = n^2$, and any $A_0(x)$ that vanishes at $x = 0$ is in Arnold's form.) The method resembles that of section 4 and also that described in [22]. However, instead of aiming at matrices $B_r(x)$ that are block-diagonal, one stipulates that

(6.3) $$B_r(x) = \sum_{j=1}^{d} \rho_{jr}(x)\Gamma_j, \quad r > 0$$

for all terms of the formal coefficient matrix $\sum_{r=0}^{\infty} B_r(x)\varepsilon^r$ obtained from (3.1) by a formal transformation of Type I. Instead of (4.10) one gets now the recursive conditions

(6.4) $$A_0(x)P_r(x) - P_r(x)A_0(x) - \sum_{j=1}^{d} \rho_{jr}(x)\Gamma_j = H_r(x), \quad r > 0 .$$

Now assume that $H_r(x)$ is already known to have, at worst, a pole of order σ_r at $x = 0$. This is true for $r = 1$, when $H_1 = -A_1$, by (4.11). Setting

(6.5) $$P_r(x) = x^{-\sigma_r}\check{P}_r(x), \quad \rho_{jr}(x) = x^{-\sigma_r}\check{\rho}_{jr}(x)$$

in (6.4) and cancelling the common factor $x^{-\sigma_r}$, one obtains an equation for $\check{P}_r(x)$ and $\check{\rho}_{jr}(x)$ which does have a solution at $x = 0$, by Lemma 6.1. By a form of the implicit function theorem for holomorphic functions (or else, by a simple matrix argument) one concludes that the equation for $\check{P}_r(x)$, $\check{\rho}_{jr}(x)$ can be satisfied by functions holomorphic in a neighborhood of $x = 0$ that does not depend on r. It follows that (6.4) has solutions $P_r(x)$, $\rho_{jr}(x)$ with poles of order σ_r, at most, at $x = 0$.

The induction with respect to r now proceeds exactly as in section 4. The orders of the poles of $P_r(x)$ and $\rho_{jr}(x)$ again satisfy the inequality (2.4) with a restraint index κ that satisfies the inequality (4.15).

As before, the result of this section will be stated as a non-restrictive assumption:

Property 6. $A_0(0)$ is nilpotent, $A_0(x)$ is in Arnold's canonical form, and

$$A_r(x) = \sum_{j=1}^{d} \rho_{jr}(x)\Gamma_j, \quad r > 0 . \tag{6.6}$$

Property 6 prescribes a special form for all the coefficient matrices in the expansion of $A(x, \varepsilon)$, unless $A_0(0) = 0$. Lemma 2.4 then permits the construction of functions $\rho_j(x, \varepsilon)$ in $\mathcal{a}$ with the corresponding asymptotic expansion

$$\rho_j(x, \varepsilon) \sim \sum_{r=0}^{\infty} \rho_{jr}(x)\varepsilon^r, \quad j = 1, 2, \ldots, d, \quad \rho_{j0}(0) = 0 . \tag{6.7}$$

The coefficient matrix $A(x, \varepsilon)$ has therefore the form

$$A(x, \varepsilon) = A_0(0) + \sum_{j=1}^{d} \rho_j(x, \varepsilon)\Gamma_j + E(x, \varepsilon) , \tag{6.8}$$

where

$$E(x, \varepsilon) \sim 0$$

in the sense of Lemma 2.4, i.e.,

$$|E(x, \varepsilon)| \le C_N(|x|^{-1/\kappa}\varepsilon)^{N+1} \tag{6.9}$$

for all N and for $x \in \mathcal{D}_\varepsilon$, $0 < \varepsilon \le \varepsilon_0$.

On the basis of the theory developed, so far, it cannot be asserted that $E(x, \varepsilon)$ may be taken as identically zero, as was proved to be possible in section 5 for the formal simplification of section 4.

§7. Shearing and Re-adjustment.

Following a technique used by Turrittin [17] and, in a form more similar to the one here, in Iwano and Sibuya [10], Iwano [6], [7], a further reduction of the problem will now be achieved by means of a combination of shearing transformations of Types II and III (see §3). This is necessary and meaningful only if

$$n > 1 , \tag{7.1}$$

since a scalar first order linear differential equation can be trivially solved. First, a few convenient symbols will be introduced:

Let

$$\check{\rho}_{jr}(x) = \sum_{\sigma=0}^{\infty} \rho_{jr\sigma} x^{\sigma}, \quad |x| \leq x_0 \tag{7.2}$$

(cf. (6.5)). Denote by $\mathcal{P}$ the set of ordered triples (j, r, σ) for which $\rho_{jr\sigma} \neq 0$. Next, call (μ_j, ν_j) the row and column numbers, respectively, of the one non-zero entry of the matrix Γ_j. Finally, define $\Omega(a)$ by

$$\Omega(a) := \mathrm{diag}(1, a, a^2, \ldots, a^{n-1}) . \tag{7.3}$$

The shearing transformation (Types II and III combined)

$$y = \Omega(\varepsilon^{\alpha} x^{\beta}) z , \tag{7.4}$$

with α and β to be suitably determined, takes the differential equation (3.1), in which $A(x, \varepsilon)$ has now the form (6.8), into

$$\varepsilon^{h-\alpha} x^{-k-\beta} \frac{dz}{dx} = B(x, \varepsilon) z , \tag{7.5}$$

where

$$B(x,\varepsilon) = A_0(0) + \sum_{j=1}^{d} \rho_j(x,\varepsilon)\varepsilon^{-\delta_j\alpha} x^{-\delta_j\beta} \Gamma_j \tag{7.6}$$
$$-\varepsilon^{h-\alpha} x^{-k-\beta-1} \operatorname{diag}(0, \beta, \ldots, (n-1)\beta) + E^*(x,\varepsilon) .$$

Here,

$$\delta_j = \mu_\nu - \nu_j + 1 \tag{7.7}$$

and $E^*(x,\varepsilon) \sim 0$. If the $\rho_j(x,\varepsilon)$ in (7.6) are replaced by their series expansions from (6.5) and (7.2) the terms of the resulting summation are

$$\rho_{jr\sigma}\varepsilon^{r-\delta_j\alpha} x^{\sigma-\sigma_r-\delta_j\beta}, \quad (j,r,\sigma) \in \mathcal{P} . \tag{7.8}$$

The triples of $\mathcal{P}$ with $\delta_j > 0$ correspond to terms of (7.8) that occur on or below the main diagonal, while all terms with $\delta_j \leqslant 0$ are above the main diagonal. The subset of $\mathcal{P}$ with $\delta_j > 0$ will be called $\mathcal{P}_L$ ("L" for "lower"). Note that, since $\delta_{j0}(0) = 0$, no triple $(j,0,0)$ is in $\mathcal{P}$.

The parameters α, β are to be chosen so that one term in (7.8) becomes a constant. For a convenient description of the proper choices for α and β the pointsets

$$u = r/\delta_j, \quad v = (\sigma - \sigma_r)/\delta_j, \quad (j,r,\sigma) \in \mathcal{P} \tag{7.9}$$

in an auxiliary (u,v)-plane is introduced. These points do not exist for $\delta_j = 0$, unless the (u,v)-plane is interpreted as a projective plane and those points are taken at infinity in the direction indicated by (7.9). For the construction that follows these points are irrelevant, in any case. The set defined by (7.9) will be called Q, and Q_L is the subset

corresponding to $\delta \geq 1$. Because of the inequality (2.4), the set Q_L lies in the closure of the sector S bounded below by the line

$$(7.10) \qquad u + \kappa v = 0$$

and to the left by the v-axis. Since $(j, 0, 0)$ is not in P, the origin of the (u, v)-plane is not in Q. Also, the fact that $|\delta_j| \leq n$ implies that neither the set Q itself, nor its projection on the coordinate axes has accumulation points in the finite plane. The set $Q - Q_L$ lies in the reflection of the sector S in the origin.

To take into account the term with the powers $\varepsilon^{h-\alpha} x^{-k-1-\beta}$ in (7.6), the point with coordinates $u = h$, $v = -k - 1$ is added to Q_L except when $\beta = 0$, and the so augmented set will be called Q_L^*. If $\kappa > h/(k + 1) > 0$, the line (7.10) must be replaced by

$$u + \frac{h}{k+1} v = 0$$

and S must be accordingly increased to a larger sector S^*.

The determination of α and β that will now be described is related to the use of the Newton-Puiseux polygon in several other papers on the asymptotic theory of differential equation; for instance in [10], [6], [11]. Let Π_L^* be the boundary polygon of the convex hull of Q_L^* and Π_U the boundary polygon of the convex hull of $Q - Q_L$. Observe that Π_L^* is never empty, since Q_L^* always contains the point $(h, -k - 1)$. However $Q - Q_L$ may exceptionally be empty.

Among the left-most vertices of Π_L^* there is one with the smallest ordinate. Let this vertex be (u_0, v_0). Two quite different cases must now be distinguished.

Case 1. $u_0 < h$.

Take $\alpha = u_0$, $\beta = v_0$.

Case 2. $u_0 = h$ ($u_0 > h$ is impossible).

Take $\alpha = h$, $\beta = 0$.

The second case will be postponed, for the time being. In the first case, there is now, indeed, at least one term in (7.8) that has both exponents equal to zero, and for which $(j, r, \sigma) \in \mathcal{P}_\ell$. Also, for all triples in $\mathcal{P}$ one has $r - \delta_j\alpha \geq 0$. It must still be shown that the new non-zero exponents of ε and x in (7.8) satisfy an inequality such as (2.4), i.e., that for a suitable positive number κ' one has

$$\sigma_j - \sigma + \delta_j\beta \leq (r - \delta_j\alpha)/\kappa' . \tag{7.11}$$

To find such a κ', observe that the point set $\tilde{Q}$:

$$\left(\frac{r}{\delta_j} - \alpha, \ \frac{\sigma - \sigma_r}{\delta_j} - \beta\right), \ (j, r, \sigma) \in \mathcal{P},$$

is obtained from the set Q by the translation $(-\alpha, -\beta)$. The point u_0, v_0 of Q_L^* becomes the origin, which is not counted, since its contribution is combined with $A_0(0)$. Therefore, there exist infinitely many straight lines through the origin with non-positive slope which separate the translate $\tilde{Q}_L$ of Q_L from the translate of $Q - Q_L$. This remains true if $\tilde{Q}_L$ is replaced by $\tilde{Q}_L^*$, the set obtained by adding the point $(h - \alpha, -k - 1 - \beta)$. Let

$$u + \kappa' v = 0 \quad \text{with} \quad 0 < \kappa' \leq \infty$$

one of those lines. The number κ' cannot exceed κ. For later use the stronger restriction

(7.12) $$\kappa' < \kappa$$

will be imposed.

For all points of $\tilde{Q}_L$ one has

$$\frac{r}{\delta_j} - \alpha + \kappa'\left(\frac{\sigma - \sigma_j}{\delta_j} - \beta\right) \geq 0, \quad \delta_j > 0 ,$$

which is equivalent with (7.11) for those points. For points in $\tilde{Q} - \tilde{Q}_L$ the number δ_j is negative. As these points lie below the line $u + \kappa' v = 0$ the inequality (7.11) is again true. It is also valid for $(h - \alpha, -k - 1 - \beta)$.

The transformed differential equation (7.5) is not yet of the form (3.1), because α and β are, in general, non-integral rational numbers with least denominator not exceeding n. By appropriate re-adjustment transformations of Types V and VI, say,

(7.13) $$x = \tilde{x}^q, \quad \varepsilon = \tilde{\varepsilon}^m ,$$

the differential equation can be restored to a form involving only integral powers of the independent variable and the parameter. The new differential equation is now

(7.14) $$\tilde{\varepsilon}^{(h-\alpha)m}\,\tilde{x}^{-(k+\beta+1)q+1}\,\frac{dz}{d\tilde{x}} = \tilde{B}(\tilde{x}, \tilde{\varepsilon})z ,$$

with

(7.15) $$\tilde{B}(\tilde{x}, \tilde{\varepsilon}) = qB(\tilde{x}^q, \tilde{\varepsilon}^m) ,$$

$B(x, \varepsilon)$ being as in (7.6). The right member of (7.6), when expressed in terms of $\tilde{x}$, $\tilde{\varepsilon}$, can be written as a power series

$$(7.16) \qquad \sum_{\tilde{r}=0} \tilde{B}_{\tilde{r}}(\tilde{x})\tilde{\varepsilon}^{\tilde{r}} ,$$

whose coefficients, thanks to the construction of the shearing transformation, have the form

$$(7.17) \qquad \tilde{B}_{\tilde{r}}(\tilde{x}) = \tilde{x}^{-\tilde{\sigma}_{\tilde{r}}} \check{\tilde{B}}_{\tilde{r}}(\tilde{x}) ,$$

where $\check{\tilde{B}}_{\tilde{r}}(\tilde{x})$ is holomorphic in $|\tilde{x}| \leq x_0'^{1/q}$ and

$$(7.18) \qquad \tilde{\sigma}_{\tilde{r}} \leq \tilde{r}/\tilde{\kappa} , \quad \tilde{\kappa} = m\kappa'/q .$$

It remains to verify that

$$(7.19) \qquad \tilde{B}(\tilde{x}, \tilde{\varepsilon}) \sim \sum_{\tilde{r}=0}^{\infty} qB_{\tilde{r}}(\tilde{x})\tilde{\varepsilon}^{\tilde{r}}$$

in the sense of the definition in (2.6), i.e., that the N^{th} partial sum $\tilde{B}^{(N)}(\tilde{x}, \tilde{\varepsilon})$ of the series in (7.19) satisfies an inequality of the form

$$(7.20) \qquad |\tilde{B}(\tilde{x}, \tilde{\varepsilon}) - \tilde{B}^{(N)}(\tilde{x}, \tilde{\varepsilon})| \leq C_N(|x|^{-1/\tilde{\kappa}}\tilde{\varepsilon})^{N+1}$$

for $\tilde{x} \in \tilde{\mathcal{S}}_{\tilde{\varepsilon}}$, $0 < \tilde{\varepsilon} \leq \varepsilon_0^{1/m}$. Here $\tilde{\mathcal{S}}_{\tilde{\varepsilon}}$ is defined by inequalities of the form

$$(7.21) \qquad \tilde{t}_0^{-1}\tilde{\varepsilon}^{\tilde{\kappa}} \leq |\tilde{x}| \leq x_0^{1/\varepsilon}, \quad |\arg \tilde{x}| \leq \delta_0/m .$$

To prove (7.20), denote by $\rho_j^{(M)}(x, \varepsilon)$ the M^{th} partial sum of the series for $\rho_j(x, \varepsilon)$ in (6.7). Then there is a constant c_M such that

$$(7.22) \qquad |\rho_j(x, \varepsilon) - \rho_j^{(M)}(x, \varepsilon)| \leq c_M(|x|^{-1/\kappa}\varepsilon)^{M+1}$$

for $1 \leq j \leq d$, $x \in \mathcal{S}_{\varepsilon}$, $0 < \varepsilon \leq \varepsilon_0$. If $B^{(M)}(x, \varepsilon)$ is the function obtained by replacing $\rho_j(x, \varepsilon)$ in the right member of (7.6) by $\rho_j^{(M)}(x, \varepsilon)$, one has, therefore

$$(7.23) \qquad |B(x, \varepsilon) - B^{(M)}(x, \varepsilon)| \leq c_M(|x|^{-1/\kappa}\varepsilon)^{M+1}\varepsilon^{-n\alpha}|x|^{-n\beta}$$

in the same domain. The integer M can be chosen so large that all terms of the partial sum $\tilde{B}^{(N)}(\tilde{x}, \tilde{\varepsilon})$ occur in $qB^{(M)}(\tilde{x}^q, \tilde{\varepsilon}^m)$. Then

$$(7.24) \quad \tilde{B}(\tilde{x}, \tilde{\varepsilon}) - \tilde{B}^{(N)}(\tilde{x}, \tilde{\varepsilon}) = q(B(\tilde{x}^q, \tilde{\varepsilon}^m) - B^{(M)}(\tilde{x}^q, \tilde{\varepsilon}^m)) + Q^{(M)}(\tilde{x}, \tilde{\varepsilon}) ,$$

where $Q^{(M)}(\tilde{x}, \tilde{\varepsilon})$ has the form

$$(7.25) \qquad Q^{(M)}(\tilde{x}, \tilde{\varepsilon}) = \sum_{\tilde{r}=N+1}^{Mm} \check{q}_{\tilde{r}}(\tilde{x})\tilde{x}^{-\tilde{\sigma}\tilde{r}}\tilde{\varepsilon}^{\tilde{r}}$$

with $\check{q}_{\tilde{r}}(\tilde{x})$ holomorphic in $|\tilde{x}| \leq x_0^{1/q}$. From (7.23), (7.24), (7.25) one gets, using also (7.18), that

$$(7.26) \quad \begin{aligned} |\tilde{B}(\tilde{x}, \tilde{\varepsilon}) - \tilde{B}^{(N)}(\tilde{x}, \tilde{\varepsilon})| &\leq qc_M(|x|^{-1/\kappa}\varepsilon)^{M+1}\varepsilon^{-n\alpha}|x|^{-n\beta} \\ &\quad + k_M(|\tilde{x}|^{-1/\tilde{\kappa}}\tilde{\varepsilon})^{N+1}, \quad (k_M \text{ a constant}) . \end{aligned}$$

This is valid for $\tilde{x} \in \tilde{\mathcal{S}}_{\tilde{\varepsilon}}$, $0 < \tilde{\varepsilon} \leq \varepsilon_0^{1/m}$, provided $\tilde{t}_0 \leq t_0^{1/q}$. Now, (7.26) can be written

$$(7.27) \quad |\tilde{B}(\tilde{x}, \tilde{\varepsilon}) - \tilde{B}^{(N)}(\tilde{x}, \tilde{\varepsilon})| \leq (|\tilde{x}|^{-1/\tilde{\kappa}}\tilde{\varepsilon})^{N+1}[k_M + qc_M|\tilde{x}|^{-A_M}\tilde{\varepsilon}^{B_M}]$$

with

$$\begin{aligned} A_M &= [\kappa^{-1}(M + 1)) + n\beta]q - \tilde{\kappa}^{-1}(N + 1) \\ B_M &= [(M + 1) - n\alpha]m - (N + 1) . \end{aligned}$$

By (7.21),

$$|\tilde{x}|^{-A_M}\tilde{\varepsilon}^{B_M} \leq (\tilde{t}_0^{-1}\tilde{\varepsilon}^{\tilde{\kappa}})^{-A_M}\tilde{\varepsilon}^{B_M} = \tilde{t}_0^{A_M}\tilde{\varepsilon}^{(M+1)m(1-\kappa'/\kappa)-nq\tilde{\kappa}\beta-nm\alpha} .$$

In the last equality the second formula of (7.18) has also been used.

Thanks to the inequality (7.12), M can be chosen so large that the exponent of $\tilde{\varepsilon}$ in the last formula is positive. With M so chosen the expression in brackets in (7.27) is bounded for $\tilde{x} \in \tilde{\mathcal{D}}_{\tilde{\varepsilon}}$, $0 < \tilde{\varepsilon} \leq \varepsilon_0^{1/m}$. This completes *the proof of* (7.20), *i.e.*, of (7.19). With the definition

$$D(\tilde{x}) = \tilde{x}^{-(k+\beta+1)q}\operatorname{diag}(0, \beta, \ldots, (n-1)\beta)$$

and a return to the notation of (3.1), i.e., replacing $\tilde{x}$, $\tilde{\varepsilon}$, $h-\alpha$, $k-\beta$, $\tilde{B}(\tilde{x}, \tilde{\varepsilon})$ by

(7.28) $$x,\ \varepsilon,\ h,\ k,\ A(x, \varepsilon) ,$$

respectively, the result of this section can be stated in the form of a non-restrictive assumption, as follows.

Property 7. *If* $h > 0$, *the coefficient matrix* $A(x, \varepsilon)$ *in (3.1) has the form*

(7.29) $$A(x, \varepsilon) = A_0(0) + \sum_{j=0}^{d} \rho_j(x, \varepsilon)\Gamma_j + D(x)\varepsilon^h + E(x, \varepsilon) .$$

Here

(7.30) $$A_0(0) = A_{01} + A_{02}$$

with A_{01} *nilpotent and in Jordan form, and* A_{02} *is a lower triangular matrix of the form*

(7.31) $$A_{02} = \sum_{j=1}^{d} a_{0j}\Gamma_j .$$

The Γ_j are the matrices defined earlier in this section. The $\rho_j(x, \varepsilon)$ are in class $\mathcal{O}$, with $\lim_{\varepsilon \to 0+} \rho_j(0, \varepsilon) = 0$, and $E(x, \varepsilon) \sim 0$. The matrix $D(x)$ is diagonal.

In general, the constant lead matrix $A_0(0)$ in (7.29) has more than one eigenvalue. The reduction process can then be continued as described in section 5. However, the possibility that $A_0(0)$ has again only one eigenvalue must also be explored. This is the subject of the next section.

Now Case 2 above, in which the shearing exponents are $\alpha = h$, $\beta = 0$, remains to be considered. If that situation arises ε disappears from the left member of (7.5). The asymptotic solution of such differential equations - in the original notation of (3.1) - can be found in section 10.

§8. An Exceptional Case: All Eigenvalues Equal After Shearing.

In this section it will be assumed that the lead matrix $A_0(0)$ in (7.29) has only one distinct eigenvalue. Then the block-diagonalization process of sections 4 and 5 is omitted from the sequence of operations in the reduction process. The steps to be performed next are: Reduction to Arnold's form, shearing, re-adjustment (from x, ε to $\tilde{x}, \tilde{\varepsilon}$), multiplication by $\tilde{\varepsilon}^{-\tilde{\alpha}}\tilde{x}^{-\tilde{\beta}}$.

Let

$$\tilde{\varepsilon}^{\tilde{h}}\tilde{x}^{-\tilde{k}}\frac{d\tilde{y}}{d\tilde{x}} = \tilde{A}(\tilde{x}, \tilde{\varepsilon})\tilde{y} \tag{8.1}$$

be the problem so obtained, and assume that the new lead matrix $\tilde{A}_0(0)$ has again only one distinct eigenvalue. Here $\tilde{A}(\tilde{x}, \tilde{\varepsilon})$ is the matrix called $\tilde{B}(\tilde{x}, \tilde{\varepsilon})$ in (7.19). It has the following structure:

$$\tilde{A}(\tilde{x}, \tilde{\varepsilon}) = q\tilde{A}_0(0) + \sum_{j=0}^{d} \tilde{\rho}_j(x, \varepsilon)\Gamma_j \, , \tag{8.2}$$

$$\tilde{A}_0(0) = A_{01} + \sum_{j=0}^{d} \kappa_j\Gamma_j \, . \tag{8.3}$$

The matrix A_{01} is as in (7.30), i.e.,

$$A_{01} = H_1 \oplus H_2 \oplus \cdots \oplus H_p \, , \tag{8.4}$$

where H_j is a "shifting matrix" of order m_j (see (6.1)). (By definition, a shifting matrix of order greater than one is one that has a line of 1s above the diagonal and zero entries elsewhere. A shifting matrix of order one is zero.) The κ_j are constants. They are zero for all j with $\mu_j < \nu_j$.

The eigenvalues of $\tilde{A}_0(0)$, which is a lower block-triangular matrix in the partition induced by the blocks in (8.4), are the eigenvalues of its diagonal blocks. By assumption, there is only one distinct eigenvalue, λ_0, present. Let T_s, $s = 1, 2, \ldots, p$, be constant $m_s \times m_s$ matrices such that a transformation of Type I with the block diagonal coefficient matrix

$$T = T_1 \oplus T_2 \oplus \cdots \oplus T_p \tag{8.5}$$

takes the diagonal blocks of $\tilde{A}_0(0)$ into their Jordan form

$$\lambda_0 I_s + H_s, \quad s = 1, 2, \ldots, p \tag{8.6}$$

(I_s is the identity matrix of order m_s). Such a transformation takes each block $\tilde{A}^{\mu\nu}(\tilde{x}, \tilde{\mu})$ of $\tilde{A}(\tilde{x}, \tilde{\varepsilon})$ into $T_\mu^{-1}\tilde{A}^{\mu\nu}(\tilde{x}, \tilde{\varepsilon})T_\nu$, so that, in particular, any block $\tilde{A}_0^{\mu\nu}(0)$ that is a zero matrix remains a zero matrix after the transformation. After that, the entries λ_0 in the main diagonal can be removed by a transformation of Type VI. The new coefficient matrix $\tilde{\tilde{A}}(\tilde{x}, \tilde{\varepsilon})$ so obtained has a lead matrix of the form

$$\tilde{\tilde{A}}_0(0) = A_{01} + \tilde{\tilde{A}}_{02}\ , \tag{8.7}$$

with A_{01} as in (8.4) and $\tilde{\tilde{A}}_{02}$ <u>strictly</u> lower blocktriangular.

Next, $\tilde{\tilde{A}}_0(0)$ is changed into its Jordan form by another transformation of Type I with a constant matrix. This is not necessary if

$$\tilde{\tilde{A}}_{02} = 0\ . \tag{8.8}$$

Otherwise, the leadmatrix is now again a blockdiagonal matrix composed

of blocks such as (8.6), but, the unique eigenvalue will generally not be the same λ_0 as in the analogous transformation of $\tilde{A}_0(0)$. Moreover, and this is decisive, a Lemma of Turrittin in [17], (proved somewhat differently in [18] as Lemma 19.4), says that not all blocks will have the same size as before in (8.6). In fact the Lemma assures that after a finite number of such sequences of transformations, the lead matrix will consist of one block only.

If (8.8), is true, (8.7) shows that $\approx{A}_0(0)$ is itself a nilpotent Jordan matrix and the next shearing can be performed immediately. No change in the block structure occurs then.

Thus, under the assumption of this section that all shearings introduce lead-matrices with only one distinct eigenvalue, one is eventually led to a problem with the additional property that all nonzero entries introduced by subsequent shearings are in the last rows of the <u>diagonal</u> blocks. If only one block is present, this is, of course, no further restriction.

The shearing, in this situation, introduces exponents α and β that are, by necessity, integers, because there must appear nonzero entries on the main diagonal of the lead matrix after the shearing, otherwise the new unique eigenvalue would have to be zero, contrary to the fact that at least one nonzero entry has been added by the shearing. Now, the shearing introduces nonzero diagonal entries only if δ_j in (7.7)

is equal to one in the definition of $(u_0, v_0) = (\alpha, \beta)$. In that case α, β are, indeed, integers.

It follows that no re-adjustment is necessary: $x = x$, $\tilde{\varepsilon} = \varepsilon$. Therefore $\tilde{h} = h - \alpha$. If $\alpha > 0$ occurs h times, at most, in the repetitions of the reduction, a problem without a power of ε in the left member has been reached and one proceeds as in section 10.

There remains now one final possibility: The shearing exponent α, in the situation described in the last two paragraphs is forever zero in each repetition of the reduction process. This case is taken care of in the lemma below.

Lemma 8.1. *Assume that the given differential equation has the form described in (8.1) through (8.4) and that $\tilde{A}_0(0)$ has no nonzero entries outside of the diagonal blocks. Assume that, in addition, the sequence of operations: Reduction to Arnold's form, eigenvalue shifting, shearing, multiplication by the appropriate power of $\tilde{x}$, always leads to a problem of the same type with the same value of $\tilde{h}$. Then the matrix $\tilde{A}_0(\tilde{x})$ has, identically in $\tilde{x}$, exactly one eigenvalue.*

Proof. To simplify the notation the tildes will be omitted from the notation. The shearing transformations, as well as the reduction to Arnold's form, subject $A_0(x)$ to similarity transformations and, therefore, do not change its eigenvalues. Let $A_0^{(m)}(x)$ be what becomes of $A_0(x)$ after m iterations of the reducing cycle, with $A_0^{(0)}(x) = A_0(x)$. If $\mu_1^{(m)}(x), \ldots, \mu_n^{(m)}(x)$ are the eigenvalues of $A_0^{(m)}(x)$, then, by assumption,

(8.9) $$\lim_{x \to 0} \mu_j^{(m)}(x) = \lambda_0^{(m)} \neq 0, \quad j = 1, \ldots, n .$$

Let β_m the shearing exponent in the m^{th} iteration. Then one has also

(8.10) $$\mu_j^{(m+1)}(x) = x^{-\beta_m}(\mu_j^{(m)}(x) - \lambda_0^{(m)}), \quad m = 0, 1, \ldots ,$$

i.e.,

$$\mu_j^{(m)}(x) = \mu_j^{(m+1)}(x)x^{\beta_m} + \lambda_0^{(m)} .$$

Hence,

$$\mu_j^{(0)}(x) = \mu_j^{(1)}(x)x^{\beta_0} + \lambda_0^{(0)} = \mu_j^{(2)}(x)x^{\beta_0+\beta_1} + \lambda_0^{(1)}x^{\beta_0} + \lambda_0^{(0)} .$$

By repetition of this procedure one proves that, for arbitrarily large ℓ,

(8.11) $$\mu_j^{(0)}(x) = \lambda_0^{(0)} + \lambda_0^{(1)}x^{\beta_0} + \cdots + \lambda_0^{(\ell)}x^{\sum_{s=0}^{\ell-1}\beta_s} + O(x^{\sum_{s=0}^{\ell}\beta_s}) .$$

On the other hand, the functions $\mu_j^{(0)}(x)$, as eigenvalues of $A_0^{(0)}(x)$, which is holomorphic at $x = 0$, possess convergent expansions in ascending fractional powers of x. By the uniqueness of power series expansions this series must be the one constructed in (8.11), i.e.,

$$\mu_j^{(0)}(x) = \sum_{m=0}^{\infty} \lambda_0^{(m)}x^{\sum_{s=0}^{m-1}\beta_s} .$$

This series is independent of j, which proves the lemma.

In the situation described by the lemma above, i.e., if $A_0(x)$ has, identically in x, exactly one eigenvalue $\lambda_0(x)$ an eigenvalue-

shifting transformation of Type VI with $\lambda(x) = \lambda_0(x)$ produces a similar problem with a new function $A_0(x)$ which is identically nilpotent.

$A_0(x)$ may already be assumed to be in Arnold's form. Therefore its diagonal blocks are shifting matrices. After the next shearing, either the new lead matrix $\tilde{A}_0(0)$ has nonzero entries in the blocks below the diagonal blocks or the nonzero entries originated from matrices $A_r(x)$ with $r > 0$, in which case $\alpha > 0$. In either case situations have now been reached which were discussed before, and the whole process must terminate after a finite number of steps.

The theorem below summarizes what has been achieved so far.

Theorem 8.1. By a finite number of transformations of Types I to VI, described in § 3, every differential equation of the form (3.1) can be reduced to a finite number of equations of the same form for which either $n = 1$ or $h = 0$.

§9. Asymptotic Solution of the Scalar Equation.

If (3.1) is a scalar equation, one of its solutions is

$$(9.1)\qquad y(x,\varepsilon) = \exp\{\varepsilon^{-h}\int_{x_0}^{x} \xi^k A(\xi,\varepsilon)d\xi\} .$$

In this section the asymptotic nature of this solution will be investigated.

When $k < 0$, the integral in (9.1) is not always in class $\mathcal{A}$. Therefore the following wider classes are introduced.

Definition 9.1. A function $A(x,\varepsilon)$ is said to be in class $\mathcal{A}^*$, if it has all the properties of class $\mathcal{A}$ except that $\check{A}_r(x)$ need be holomorphic in

$$0 < x \le x, \quad |\arg x| \le \delta_0$$

only. But $\check{A}_r(x)$ must be bounded in that domain. A function $A(x,\varepsilon)$ such that $x^p A(x,\varepsilon) \in \mathcal{A}^*$ for some $p > 0$ is said to belong to class $\mathcal{A}^{**}$.

The asymptotic relation (2.2) is defined in these wider classes exactly as in Definition 2.1, and the lemmas of Section 2 remain valid in them.

The integrals

$$(9.2)\qquad B_r(x) = \int_{x_0}^{x} \xi^k A_r(\xi)d\xi ,$$

with $A_r(x)$ as in (2.2), (2.3) have convergent ascending series in powers of x, except for the possible occurrence of a logarithmic term. The series may begin with a negative power of x.

Lemma 9.1. Let

$$(9.3)\qquad Q(x,\varepsilon) = \sum_{r=0}^{h} B_r(x)\varepsilon^{r-h} ,$$

and

$$F(x,\varepsilon) = \varepsilon^{-h} \int_{x_0}^{x} \xi^k A(\xi,\varepsilon)\,d\xi - Q(x,\varepsilon) . \tag{9.4}$$

Then $F(x,\varepsilon) \in \mathcal{A}^*$, and

$$F(x,\varepsilon) \sim \sum_{r=h+1}^{\infty} B_r(x)\varepsilon^{r-h} . \tag{9.5}$$

Proof. It is convenient to re-write the series in (9.5) as $\sum_{r=0}^{\infty} C_r(x)\varepsilon^r$ with

$$C_0(x) \equiv 0;\ C_r(x) = B_{r+h}(x), \quad r > 0 . \tag{9.6}$$

Then

$$|C_r(x)| \le \begin{cases} \text{const.} , & \text{for } k+1-\sigma_{r+h} > 0 \\ \text{const.}\ |x|^{k+1-\sigma_{r+h}}, & \text{for } k+1-\sigma_{r+h} < 0 \\ \text{const.}\ |\log x| , & \text{for } k+1-\sigma_{r+h} = 0 , \end{cases} \tag{9.7}$$

because of (9.2) and (2.3). As $C_0(x) \equiv 0$, the inequalities (2.4) and (9.7) imply the relations

$$C_r(x) = \check{C}_r(x) x^{-\check{\sigma}_r}, \quad \check{C}_r(x) \text{ bounded} \tag{9.8}$$

$$\check{\sigma}_r < r/\check{\kappa}, \quad 0 < \check{\kappa} \le \kappa \le \infty . \tag{9.9}$$

If $\kappa \le h/(k+1)$ one has $\check{\kappa} = \kappa$.

To prove (9.5), recall that by (2.6),

$$A(x,\varepsilon) = \sum_{r=0}^{N} A_r(x)\varepsilon^r + (x^{-1/\kappa}\varepsilon)^{N+1} R_N(x,\varepsilon) , \tag{9.10}$$

with $R_N(x, \varepsilon)$ uniformly bounded in $\mathcal{S}_\varepsilon$, for $0 < \varepsilon \leq \varepsilon_0$. Let $M > N > 0$, replace N by M in (9.10) and insert it into (9.4). Referring to (9.2) and (9.6) one then obtains the relation

$$F(x, \varepsilon) = \sum_{r=0}^{N} C_r(x)\varepsilon^r + \sum_{r=N+1}^{M} C_r(x)\varepsilon^r$$

$$(9.11) \qquad + \varepsilon^{M+1-h} \int_{x_0}^{x} R_M(\xi, \varepsilon)\xi^{k-(M+1)/\kappa} d\xi \,.$$

From (9.8) and (9.9) one concludes that the second summation in the right hand member of (9.11) satisfies the inequality

$$(9.12) \qquad \left| \sum_{r=N+1}^{M} C_r(x)\varepsilon^r \right| \leq \text{const.} \; |x^{-1/\check{\kappa}}\varepsilon|^{N+1} \,.$$

If M is so large that $k - (M + 1)/\kappa < -1$, the last terms in (9.11) is less than

$$(9.13) \qquad \text{const.} \; \varepsilon^{M+1-h} |x|^{k+1-(M+1)/\kappa}$$

in modulus, i.e., less than

$$(9.14) \qquad \text{const.} \; \varepsilon^{M+1-h-(N+1)} |x|^{k+1-(M+1)/\kappa+(N+1)/\check{\kappa}} |x^{-1/\check{\kappa}}\varepsilon|^{N+1} \,.$$

One verifies immediately that for

$$t_0^{-1}\varepsilon^{\check{\kappa}} \leq |x| \leq x_0$$

the last expression is $O(|x^{-1/\check{\kappa}}\varepsilon|^{N+1})$, provided M is taken so large that the exponent $k + 1 - (M + 1)/\kappa + (N + 1)/\check{\kappa}$ in (9.14) is negative. Insertion of these estimates into (9.11) shows that

$$F(x, \varepsilon) = \sum_{r=0}^{N} C_r(x)\varepsilon^r + O(|x^{-1/\check{\kappa}}\varepsilon|^{N+1}),$$

for $0 < \varepsilon \leq \varepsilon_0$, $x \in \mathcal{S}_\varepsilon$, which completes the proof of the lemma.

From the expansion

$$F(x) \sim \sum_{r=0}^{\infty} C_r(x)\varepsilon^r, \quad C_0(x) \equiv 0,$$

which is the same as (9.5) in simpler notation, one proves, just as for ordinary asymptotic power series, that $\tilde{Y}(x, \varepsilon) := \exp(F(x, \varepsilon))$ also is in $\mathcal{A}^*$.

Thus, the theorem below has been proved.

Theorem 9.1. For $n = 1$, the differential equation (3.1) has a solution of the form

$$Y(x, \varepsilon) = \tilde{Y}(x, \varepsilon)e^{Q(x, \varepsilon)} \tag{9.15}$$

with $\tilde{Y}(x, \varepsilon) \in \mathcal{A}^*$ and $Q(x, \varepsilon)$ a polynomial of degree h in ε^{-1}, as described in (9.2), (9.3).

Remark. It is easy to analyze the function $Q(x, \varepsilon)$ in more detail. This will not be elaborated upon here. Since the result depends decisively on whether $k > -1$, $= -1$ or < -1 the description is a little tedious. In all cases the first term, $B_0(x)\varepsilon^{-h}$, dominates the others in $\mathcal{S}_\varepsilon$, at least for large t_0.

§ 10. Asymptotic Solution when $h = 0$.

The problem is to study the solutions of a system of the form

$$x^{-k}\frac{dy}{dx} = A(x,\varepsilon)y, \quad A(x,\varepsilon) \in \mathcal{A} \tag{10.1}$$

for small ε in the region $\mathcal{S}_\varepsilon$. Although the character of the solution depends considerably on whether $k > -1$, $k = -1$ or $k < -1$, the discussion will be unified as much as possible.

Formally, (10.1) always admits matrix solutions of the form $\sum_{r=0}^{\infty} Y_r(x)\varepsilon^r$. The $Y_r(x)$ must only satisfy the differential equations

$$x^{-k}\frac{dY_r}{dx} = A_0(x)Y_r + \sum_{\substack{\mu+\nu=r \\ \mu>0}} A_\mu(x)Y_\nu(x), \quad r = 0,1,\ldots . \tag{10.2}$$

For $r = 0$, that equation is

$$x^{-k}\frac{dY_0}{dx} = A_0(x)Y_0 . \tag{10.3}$$

The nature of its solutions near $x = 0$ is described by the next lemma.

Lemma 10.1. The differential equation (10.3) possesses a particular fundamental matrix solution of the form

$$Y_0(x) = \tilde{Y}_0(x)x^G e^{Q(x)} \tag{10.4}$$

with the following properties.

(i) $\tilde{Y}_0(x)$ has an asymptotic series in powers of $x^{1/p}$, as $x \to 0$ in $|\arg x| \le \delta_0$.

$$\det \tilde{Y}_0(x) \neq 0 \ \text{for}\ x \neq 0 \ \text{and}\ \tilde{Y}_0(0) \not\equiv 0 ,$$

but $\det \tilde{Y}_0(0)$ may be zero. The number p is a positive integer.

(ii) $Q(x)$ <u>is a diagonal matrix</u>. <u>If it is not identically zero, it is a polynomial in</u> $x^{-1/p}$ <u>without constant term</u>.

(iii) G <u>is a constant matrix, which commutes with</u> $Q(x)$.

(iv) <u>If</u> $k = -1$, <u>then</u> $p = 1$, $Q(x) \equiv 0$ <u>and</u> $\tilde{Y}_0(x)$ <u>is holomorphic at</u> $x = 0$. <u>If</u> $k \geq 0$, <u>then</u> $G = 0$, <u>as well, and</u> $\det \tilde{Y}_0(0) \neq 0$.

This lemma summarizes well known classical results which can, e.g., be found in [18] Chapters I-V.

Two preliminary transformation will now be performed. While not necessary, they are helpful in that they simplify the subsequent analysis. First, set

$$\tilde{x} = x^{1/p} \tag{10.5}$$

in (10.1). This produces a differential equation in which k is replaced by $p(k+1) - 1$ and p becomes equal to 1. As often before in this paper, the notation will be changed back to the original one. Then the non-restrictive condition below can now be imposed.

<u>Property 8</u>. <u>In Lemma</u> 10.1, <u>one has</u> $p = 1$.

Next, set

$$T(x) = \tilde{Y}_0(x) x^G , \tag{10.6}$$

for abbreviation, and perform the transformation

$$y = T(x) z \tag{10.7}$$

in (10.1). There exist real numbers g_1, g_2 (they are not unique) such that

$$T(x) x^{g_1}, \ T^{-1}(x) x^{g_2} \quad \text{are bounded in} \quad 0 < |x| \leq x_0, \ |\arg x| < \delta_0 . \tag{10.8}$$

The differential equation (10.1) is now changed into

(10.9) $$x^{-k}\frac{dz}{dx} = \tilde{B}(x,\varepsilon)z \ .$$

Formally, the series in the expansion

(10.10) $$A(x,\varepsilon) \sim \sum_{r=0}^{\infty} A_r(x)\varepsilon^r, \quad \varepsilon \to 0+ \ ,$$

becomes $\sum_{r=0}^{\infty} \tilde{B}_r(x)\varepsilon^r$, with

(10.11) $$\tilde{B}_r(x) = T^{-1}(x)A_r(x)T(x), \ \ r > 0 \ .$$

The new leading coefficient $\tilde{B}_0(x)$ is more easily calculated indirectly: In analogy to the formal series solution $\sum_{r=0}^{\infty} Y_r(x)\varepsilon^r$ the differential equation *(10.9) has a series solution* $\sum_{r=0}^{\infty} Z_r(x)\varepsilon^r$ *with* $Y_r(x) = T(x)Z_r(x)$. In particular, therefore, (10.4) implies

(10.12) $$Z_0(x) = e^{Q(x)} \ .$$

This must be a solution of the equation

(10.13) $$x^{-k}\frac{dZ_0}{dx} = \tilde{B}_0(x)Z_0 \ .$$

Therefore,

(10.14) $$\tilde{B}_0(x) = x^{-k}\frac{dZ_0}{dx}Z_0^{-1} = x^{-k}Q'(x) \ .$$

In explicit form, if $Q(x) \not\equiv 0$, then

(10.15) $$Q(x) = \sum_{j=1}^{m} Q_j x^{-j}, \ \ Q_m \neq 0 \ ,$$

so that

$$(10.16) \qquad \tilde{B}_0(x) = -\sum_{j=1}^{m} jQ_j x^{-j-1-k} = -x^{-k-m-1}\sum_{j=1}^{m} jQ_j x^{m-j} ,$$

unless $Q(x) \equiv 0$, when $\tilde{B}_0(x) \equiv 0$.

Returning to (10.11), one sees that by virtue of (10.8) the matrices

$$(10.17) \qquad \tilde{B}_r(x)x^{g_1+g_2+\sigma_r}$$

are bounded in $0 < |x| \le x_0$, $|\arg x| < \delta_0$ although not necessarily holomorphic at $x = 0$. Now, set

$$(10.18) \qquad B(x,\varepsilon) = \tilde{B}(x,\varepsilon)x^{k+m-1}, \quad B_r(x) = \tilde{B}_r(x)x^{k+m-1} .$$

The differential equation (10.9) can then be written

$$(10.19) \qquad x^{m+1}\frac{dz}{dx} = B(x,\varepsilon)z ,$$

and the $B_r(x)$ satisfy an order of magnitude inequality of the form:

$$(10.20) \qquad B_r(x)x^{\rho_r} \text{ is bounded for } 0 < |x| \le x_0, \ |\arg x| \le \delta_0 ;$$

$$(10.21) \qquad \rho_r \le r/\kappa_1, \quad 0 < \kappa_1 \le \infty .$$

The proof that $B(x,\varepsilon) \in \mathcal{A}^*$, i.e., that

$$(10.22) \qquad B(x,\varepsilon) \sim \sum_{r=0}^{\infty} B_r(x)\varepsilon^r$$

in the sense of Definition 2.1, resembles so much analogous previous arguments, e.g. in section 7, that a repetition is not necessary.

Thus the following lemma has been obtained.

Lemma 10.2. The transformation (10.7) (where $T(x)$ is defined by (10.6) and (10.4), takes the differential equation (10.1) (which is supposed to have Property 8) into

$$x^{m+1}\frac{dz}{dx} = B(x,\varepsilon)z\,, \tag{10.23}$$

where $B(x,\varepsilon) \in \mathcal{A}^{*}$. The leading term $B_0(x)$ in the asymptotic series (10.22) for $B(x,\varepsilon)$ is either zero, and then $m = 0$, or

$$B_0(x) = -\sum_{j=1}^{m} jQ_j x^{m-j}\,, \tag{10.24}$$

(with Q_j as in (10.15).)

The coefficients $Z_r(x)$ of the series $\sum_{r=0}^{\infty} Z_r(x)\varepsilon^r$ which solves (10.23) formally must be solutions of the differential equations

$$x^{m+1}\frac{dZ_r}{dx} = B_0(x)Z_r + \sum_{\substack{\mu+\nu=r\\ \mu>0}} B_\mu(x)Z_\nu(x), \quad r = 0,1,\ldots \tag{10.25}$$

which is the analog of (10.2).

The asymptotic calculation of the $Z_r(x)$ is based on the lemma below.

Lemma 10.3. Let $q(t)$ be a scalar polynomial $q(t) = q_1t + q_2t^2 + \cdots + q_mt^m$, $m \geq 1$, $q_m \neq 0$, and let $g(t)$ be a scalar function holomorphic and bounded in the region

$$S = \{t \,|\, t_0 \leq |t|,\ |\arg(t-\theta_0)| \leq \delta_0\}\,.$$

Then the function

$$f(t) = e^{q(t)}g(t)t^h \quad (h \text{ an integer})$$

has an antiderivative $F(t)$ of the form

$$F(t) = e^{q(t)}G(t)t^{h-m+1}$$

with $G(t)$ holomorphic and bounded in some subsector

$$S^* = \{t \mid t_1 \leq |t|, \ |\arg(t - \theta_0)| \leq \delta_1\}$$

of S. ($G(t)$ has nothing to do with the matrix G in (10.4).)

Proof. Choose $t_1 \geq t_0$ so large that the mapping $s = q(t)$ has a holomorphic inverse in the subsector of S with $|t| \geq t_1$. Define $F(t)$ by the integral

$$F(t) = \int_{\gamma(t)} e^{q(\tau)}g(\tau)\tau^h d\tau .$$

The path $\gamma(t)$, which must end at t is to be suitably chosen later. Since $t = q_m^{-1/m}s^{1/m}(1 + O(s^{-1/m}))$, as $s \to \infty$, the transformation $q(\tau) = \sigma$ changes the integral into

$$\hat{F}(s) = \hat{F}(q(t)) = F(t) = \int_{\hat{\gamma}(s)} e^{\sigma}\hat{g}(\sigma)\sigma^{(h-m+1)/m}d\sigma .$$

$\hat{\gamma}(s)$ is, of course, the image of the path $\gamma(t)$ with $\hat{g}(\sigma)$ bounded and holomorphic in the image $\hat{S} = q(S)$ of S. When written as

$$\hat{F}(s)e^{-\sigma} = \int_{\hat{\gamma}(s)} e^{\sigma-s}\hat{g}(\sigma)\sigma^{(h-m+1)/m}d\sigma , \tag{10.26}$$

the integral takes on a form dealt with in Lemma 14.2 of [18], where the setting is more general, except that the exponent that corresponds to the exponent of σ in the integral above is assumed to be non-positive.

This assumption is, however, nowhere necessary in the proof. Instead of repeating the rather lengthy arguments of that proof it will suffice to state the conclusion which is that the integral in (10.26) is $O(s^{(h-m+1)/m})$, as $s \to \infty$ in $q(s^*)$, if δ_1 is taken small enough. This completes the proof of Lemma 10.3.

Lemma 10.4. *The differential equations (B.25) possess solutions of the form*

$$Z_r(x) = \tilde{Z}_r(x)e^{Q(x)}, \tag{10.27}$$

$$\tilde{Z}_r(x) = \check{Z}_r(x)x^{-\sigma_r}, \tag{10.28}$$

where $\check{Z}_r(x)$ *is holomorphic and bounded in* $0 < |x| \leq x_0$, $|\arg x| \leq \delta_0$, *and*

$$\sigma_r \leq r/\kappa \tag{10.29}$$

for some number κ *in* $0 < \kappa \leq \infty$.

Proof. Formula (10.12) establishes the lemma for $r = 0$. For $r > 0$, replace (10.25) by the equivalent integral representation of the Z_r

$$Z_r = \int^x Z_0(x)Z_0^{-1}(\xi) \sum_{\substack{\mu+\nu=r \\ \mu>0}} B_\mu(\xi)Z_\nu(\xi)\xi^{-m-1}d\xi . \tag{10.30}$$

The integral sign is to be interpreted as indicating an as yet unspecified antiderivative taken at $\xi = x$. Assume the lemma to have been proved for $Z_0, Z_1, \ldots, Z_{r-1}$. By making use of this hypothesis, as well as formulas (10.20), (10.21) and (10.12) one finds, after multiplication with $e^{-Q(x)}$ on both sides of the equation, that (10.30) can be written in the form

$$\tilde{Z}_r = \int^x e^{Q(x)-Q(\xi)}\check{H}_r(\xi)e^{Q(\xi)-Q(x)}\xi^{-h_r}d\xi . \tag{10.31}$$

A short calculation shows that if h_r satisfies the inequality

$$h_r < \frac{r}{\kappa} \tag{10.32}$$

where κ has been taken so small that

$$\kappa \le \kappa_1/[1 + (m+1)\kappa_1] ,$$

with κ_1 as in (10.21), then $\check{H}_r(x)$ is bounded, as $x \to 0$ in $|\arg x| \le \delta_0$.

The right member of (10.31) is a matrix of n^2 scalar integrals of the form

$$\int^x e^{q_{jk}(x)-q_{jk}(\xi)} \check{h}_{jk}(\xi)\xi^{-r/\kappa} d\xi, \qquad j, k = 1, \ldots, n . \tag{10.33}$$

The $q_{jk}(x)$ are the differences of the polynomials in $1/x$ that appear as the diagonal entries of $Q(x)$. Their degrees may therefore be any integer between 0 and n. If the degree of $q_{jk}(x)$ is positive, Lemma 10.3 can be applied, after the transformation $\xi = 1/\tau$. The result, in terms of x is then, that the integral is $O(x^{2-r/\kappa})$, as $x \to 0$. If $q_{jk}(x)$ is constant, an elementary argument shows that the integral is $O(x^{1-r/\kappa})$, except if $r/\kappa = 1$, which can be avoided, by not choosing κ as an integer. Thus, the relation $\tilde{Z}_r(x) = O(x^{-r/\kappa})$, as $x \to 0$ in $|\arg x| \le \delta_0$ is always true, and the lemma is proved.

The matrices

$$Y_r(x) = T(x)Z_r(x), \quad r = 0, 1, \ldots \tag{10.34}$$

solve the differential equation (10.2). It remains to be shown that the

formal series solution $\sum_{r=0}^{\infty} Y_r(x)\varepsilon^r$ represents asymptotically a solution of (10.1). It is convenient to prove first the analogous statement for the series $\sum_{r=0}^{\infty} Z_r(x)\varepsilon^r$ with respect to the equation (10.23). The return to (10.1) is then a simple matter.

With the notation of Lemma 10.4 let $\tilde{\tilde{Z}}$ be a function in $\mathcal{A}^*$ with the asymptotic expansion

(10.35) $$\tilde{\tilde{Z}}(x,\varepsilon) \sim \sum_{r=0}^{\infty} \tilde{Z}_r(x)\varepsilon^r, \quad \text{as} \quad \varepsilon \to 0+,$$

for $x \in \mathcal{S}_\varepsilon$. Such a function exists according to Lemma 2.4, which remains literally valid in $\mathcal{A}^*$ instead of $\mathcal{A}$.

Since $\sum_{r=0}^{\infty} \tilde{Z}_r(x)e^{Q(x)}\varepsilon^r$ is, by construction, a formal solution of (10.23), it follows that

(10.36) $$\left(x^{m+1}\frac{d}{dx} - B(x,\varepsilon)\right)\left(\tilde{\tilde{Z}}(x,\varepsilon)e^{Q(x)}\right) = \Psi(x,\varepsilon)e^{Q(x)},$$

where $\Psi(x,\varepsilon) \sim 0$, i.e.,

(10.37) $$\Psi(x,\varepsilon) \leq c_N(|x|^{-1/\kappa}\varepsilon)^{N+1},$$

for $x \in \mathcal{S}_\varepsilon$, $0 < \varepsilon \leq \varepsilon_0$ and for all $N \geq 0$. Therefore, if (10.23) is converted into a differential equation for the matrix-valued function V by inserting

(10.38) $$z = (V + \tilde{\tilde{Z}}(x,\varepsilon))e^{Q(x)}$$

into it, one gets, with the help of (10.15), (10.24) and (10.36), the

differential equation

$$x^{m+1}\frac{dV}{dx} = B_0V - VB_0 + (B - B_0)V - \Psi(x, \varepsilon) , \tag{10.39}$$

and the task is now to show that it has a solution that is asymptotic to zero. To achieve this, one observes that, according to a version of the variation of parameters formula derived in [18], § 30.1, every solution of the integral equation

$$V(x, \varepsilon) = \int_{\Gamma(x)} e^{Q(x)-Q(\xi)}[(B(\xi, \varepsilon) - B_0(\xi))V(\xi, \varepsilon) - \Psi(x, \varepsilon)]\xi^{-m-1}e^{Q(\xi)-Q(x)}d\xi \tag{10.40}$$

also solves (10.39). Here, formulas (10.15) and (10.24) have again been used.

The estimation of the integral operator in the right member of (10.40) begins with the observation that

$$|B(\xi, \varepsilon) - B_0(\xi)| \leq c_0|\xi|^{-1/\kappa}\varepsilon ,$$

for $\xi \in \mathcal{D}_\varepsilon$, $0 < \varepsilon \leq \varepsilon_0$, in consequence of (2.6). Now, replace κ by a smaller number κ^*, $0 < \kappa^* < \kappa$, such that

$$\kappa^* < \frac{\kappa}{1 + \kappa(m + 1)} ,$$

and write $\mathcal{D}_\varepsilon^*$ for the subset of $\mathcal{D}_\varepsilon$ obtained by this change in the definition of $\mathcal{D}_\varepsilon$. Then

$$|B(\xi, \varepsilon) - B_0(\xi)||\xi|^{-m-1} \leq c_0t_0^{1/\kappa+m+1}\varepsilon^{1-\kappa^*(1/\kappa+m+1)} \tag{10.41}$$

for $x \in \mathcal{D}^*$, $0 < \varepsilon \leq \varepsilon_0$, and the right member can be made as small as one pleases by reducing ε_0 or t_0.

Let the $q_{jk}(x)$, $j, k = 1, 2, \ldots, n$ be defined as in (10.33), then the set of n^2 paths of integration, each ending at $\xi = x$ must be such that in each scalar integral in (10.40) the function $\operatorname{Re} q_{jk}(\xi)$ is non-increasing. The proof that this can be done for $x \in \mathcal{S}_\varepsilon^*$, provided t_0, δ_0, ε_0, x_0 are chosen small enough is a standard argument that will be omitted.

A solution V of (10.40) which is asymptotic to zero can now be constructed by the usual Picard iteration: Set $V_0 = 0$, then

$$V_1(x, \varepsilon) = -\int_{\Gamma(x)} e^{Q(x)-Q(\xi)} \cdot \Psi(\xi, \varepsilon)\xi^{-m-1} e^{Q(\xi)-Q(x)} d\xi \ .$$

The path of integration has finite length, the exponential factors are bounded for the appropriate choice of the matrix of paths $\Gamma(x)$, and (10.37) implies

$$|\Psi(\xi, \varepsilon)\xi^{-m-1}| \leq C_N^*(|x|^{-1/\kappa^*}\varepsilon)^{N+1}$$

for $x \in \mathcal{S}_\varepsilon^*$, $0 < \varepsilon \leq \varepsilon_0$ and all $N \geq 0$. (C_N^* is a constant). Hence, also

(10.42) $$|V_1(x, \varepsilon)| \leq C_N(|x|^{-1/\kappa^*}\varepsilon)^{N+1}$$

with some constant C_N.

The details of the proof from here on - whether based on the Contraction Theorem in Banach Space or explicitly on the Picard iteration argument - are analogous to those found, with variations, in almost every paper on the asymptotic nature of formal series solutions for differential equations. It suffices to point out that the possibility of making the right member of (10.41) small is the basis of the existence and

uniqueness proof for the solution of the integral equation (10.40), while (10.42) implies that this solution is small of the same order, i.e., that

$$V(x, \varepsilon) \sim 0 \quad \text{for} \quad x \in \mathcal{S}_{\varepsilon}^{*}, \quad 0 < \varepsilon \leq \varepsilon_0 , \tag{10.43}$$

in the sense of Definition (2.6).

Finally, the last result must be translated into the equivalent statement for the differential equation (10.1) by the transformation (10.7), replacement of x by $\tilde{x}$ in the notation, and return to x by the inverse of the change of variables (10.5). The proof of the theorem below is now complete.

Theorem 10.1. The differential equation (10.1):

$$x^{-k} \frac{dy}{dx} = A(x, \varepsilon)y, \quad A(x, \varepsilon) \in \mathcal{A},$$

with k an integer - positive, negative or zero - has a fundamental matrix solution of the form

$$Y(x, \varepsilon) = \tilde{Y}(x, \varepsilon)x^G e^{Q(x)} \tag{10.43}$$

with the following properties.

(i) G and $Q(x)$ are the matrices defined in Lemma 10.1;

(ii) $\tilde{Y}(x, \varepsilon) = \tilde{Y}_0(x)\tilde{\tilde{Y}}(x, \varepsilon)$, where $\tilde{Y}_0(x)$ was described in Lemma 10.1, and $\tilde{\tilde{Y}}(x, \varepsilon) \in \mathcal{A}^*$,

$$\lim_{\varepsilon \to 0+} \tilde{\tilde{Y}}(x, \varepsilon) = I .$$

Remark. The product $\tilde{Y}(x, \varepsilon)x^G$ is, of course, in class $\mathcal{A}^{**}$. Only this weaker fact is needed for the Main Theorem stated in the Introduction.

§ 11. *The Main Theorem*.

The several successive transformations of the original differential equation (1.1) can be assembled into one transformation

$$y = T(\tilde{x}, \tilde{\varepsilon})z, \quad x = \tilde{x}^m, \quad \varepsilon = \tilde{\varepsilon}^{\ell} . \tag{11.1}$$

$T(x, \varepsilon)$ is a product of matrices each corresponding to a transformation of Type I, II, III or VI. Whenever one of these transformations is applied to one of the blocks of the block diagonalization reached at that stage it can be extended to a transformation of the whole space which leaves the remaining components unchanged. Observe that the transformations of Type VI are scalar in each of the blocks of all subsequent transformations. Thus, these transformations can all be moved to the rightmost positions. Therefore, $T(\tilde{x}, \tilde{\varepsilon})$ in (11.1) has the form

$$T(\tilde{x}, \tilde{y}) = \tilde{T}(\tilde{x}, \tilde{\varepsilon})\exp\{\tilde{\varepsilon}^{-K}Q_1(\tilde{x}, \tilde{\varepsilon})\} \tag{11.2}$$

with the same structural properties as formula (1.3).

The differential equation for z resulting from transformation (11.1) consists of scalar uncoupled equations with solutions of the form described in Theorem 9.1 and, possibly, of systems without a power of ε multiplying the derivative. These systems were solved in section 10. The solutions so obtained can be assembled into a matrix $Z(\tilde{x}, \tilde{\varepsilon})$ of the same form as (11.2). Moreover, the exponential factor in (11.2) commutes with Z, since the diagonal matrix $Q_1(\tilde{x}, \tilde{\varepsilon})$ is scalar in each diagonal block of Z. Therefore the exponential factors in the product $Y = TZ$ can be

combined, which completes the proof of the Main Theorem as stated in the Introduction.

The equations of the form (1.1) for which the outer solution, as described in Theorem 1.1 has been explicitly connected with an interior or intermediate solution (see, e.g. [12], [13], [14], [19], [22]) all possess, to my knowledge, the special property that

$$|\tilde{\varepsilon}^{-H} Q(\tilde{x}, \tilde{\varepsilon})| \leq K \tag{11.3}$$

for some constant K and for certain values of $\tilde{x}$ and $\tilde{\varepsilon}$ that lie in the domain $\mathcal{D}_{\tilde{\varepsilon}}$, $0 < \tilde{\varepsilon} \leq \tilde{\varepsilon}_0$ in which the asymptotic results of Theorem 1.1 are valid. This is by no means always the case, even if $Q \in \mathcal{A}^*$, as will be illustrated by the example below. That example also shows that the methods of this paper, while constructional in principle, involve long computations, even for simple equations.

<u>Examples</u>.

The equation

$$\varepsilon^2 \frac{dy}{dx} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon & 0 & x^m \end{pmatrix} y, \quad m \text{ a positive integer }, \tag{11.4}$$

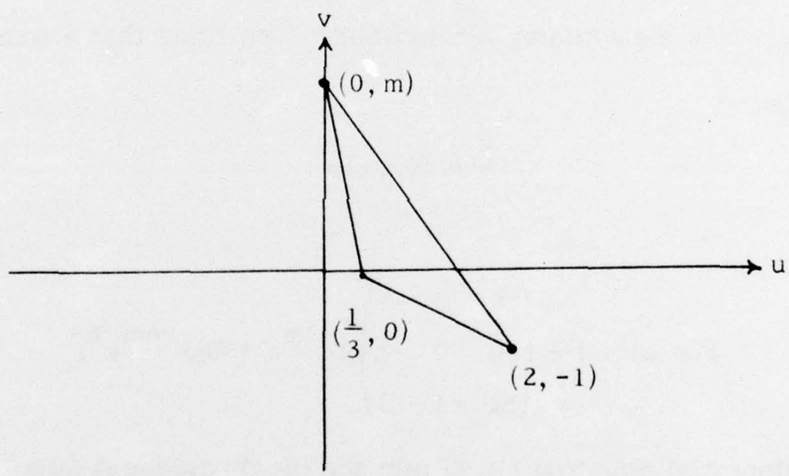
is of the type (1.1) with $h = 2$, and

$$A(x, \varepsilon) = A_0(x) + A_1(x)\varepsilon = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & x^m \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \varepsilon , \tag{11.5}$$

while the coefficient matrix is a simple companion matrix, the turning

point at $x = 0$ is relatively complicated. Two eigenvalues of $A_0(x)$ are identically zero, and the third eigenvalue becomes equal to the other two at $x = 0$. The elementary divisors of $A_0(x)$ have degrees 2 and 1 for $x \neq 0$, but for $x = 0$ there is only one elementary divisor, of degree 3.

As $A_0(0)$ is already in Jordan form and nilpotent, the decomposition of the system begins with the shearing described in §7. The figure below shows the Puiseux polygon for equation (11.4).



The appropriate shearing transformation is

$$y = \mathrm{diag}(1, x^m, x^{2m})\tilde{y} \,, \tag{11.6}$$

and the transformed differential equation becomes

$$\varepsilon^2 x^{-m} \frac{d\tilde{y}}{dx} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} x^{-3m}\varepsilon + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -m & 0 \\ 0 & 0 & -2m \end{pmatrix} x^{-m-1}\varepsilon^2 \right\} \tilde{y} \,. \tag{11.7}$$

Next, the transformation

$$(11.8)\qquad \tilde{y} = T\tilde{\tilde{y}} \quad \text{with } T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

block-diagonalizes the new lead matrix. One gets

$$(11.9)\quad \varepsilon^2 x^{-m} \frac{d\tilde{\tilde{y}}}{dx} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 & -1 \\ -1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix} x^{-3m}\varepsilon + \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} m x^{-m-1}\varepsilon^2 \right\} \tilde{\tilde{y}} .$$

The formal block diagonalization of the whole coefficient matrix by the technique of § 4 introduces a non-terminating series in powers of ε. The calculations, while elementary, are tedious. One finds that a transformation of the form

$$(11.10)\qquad \tilde{\tilde{y}} = P(x, \varepsilon) z$$

with

$$(11.11)\qquad P(x, \varepsilon) = I + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 0 \end{pmatrix} x^{-3m}\varepsilon + O(x^{-6m}\varepsilon^2)$$

takes the differential equation (11.9) into the block-diagonal form

$$(11.12)\quad \varepsilon^2 x^{-m} \frac{dz}{dx} = \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} x^{-3m}\varepsilon \right.$$

$$\left. + \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 - mx^{5m-1} & \\ 0 & 0 & -1 - 2mx^{5m-1} \end{pmatrix} x^{-6m}\varepsilon^2 + O(x^{-9m}\varepsilon^3) \right\} z ,$$

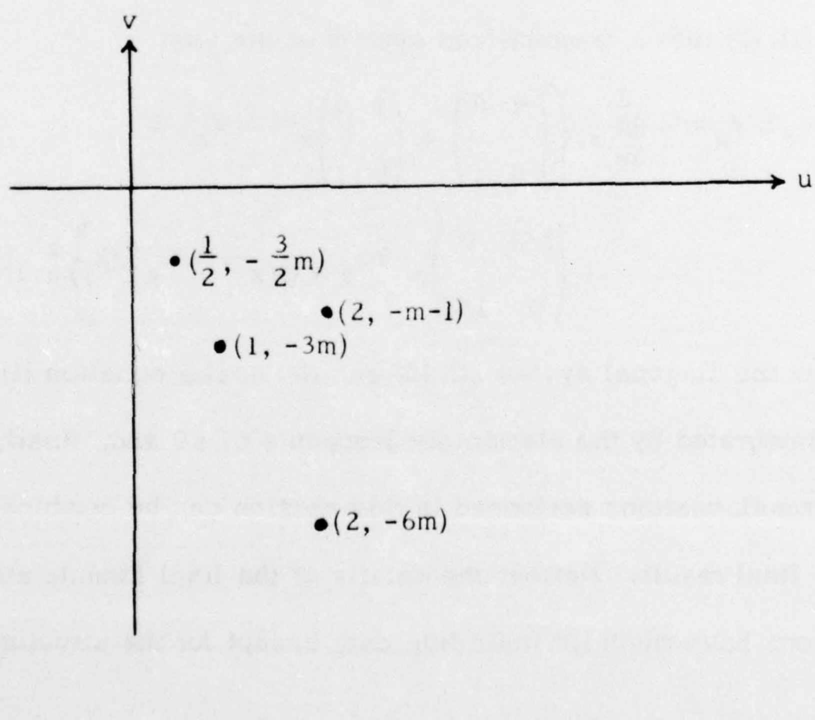
which splits into the two uncoupled problems

$$(11.13)\quad \varepsilon^2 x^{-m}\frac{du}{dx} = \left\{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + \begin{pmatrix}-1 & 0\\ -1 & 0\end{pmatrix}x^{-3m}\varepsilon + \begin{pmatrix}1 & 1\\ 1 & 1-mx^{5m-1}\end{pmatrix}x^{-6m}\varepsilon^2 + O(x^{-9m}\varepsilon^3)\right\}u$$

$$(11.14)\quad \varepsilon^2 x^{-m}\frac{dv}{dx} = \{1 + x^{-3m}\varepsilon + (-1 - 2mx^{5m-1})x^{-6m}\varepsilon^2 + O(x^{-9m}\varepsilon^3)\}v\,.$$

Here $z = (u,v)^T$.

First, the analysis of (11.13) will be continued. As the reduction to Arnold's form according to § 6 is not essential here, it will be skipped. Another shearing is then in order. Below, the points of the Puiseux diagram which can be recognized from the partial sum in (11.13) are plotted. From the form of the series it is clear that the remaining points are irrelevant for the exponents of this shearing.



The indicated shearing transformation for (11.13) is

$$(11.15) \qquad u = \operatorname{diag}(1, \ \varepsilon^{1/2} x^{-3m/2}) \tilde{u} \ .$$

It produces the new differential equation

$$(11.16) \quad \varepsilon^2 x^{\frac{m}{2}} \frac{d\tilde{u}}{dx} = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x^{-\frac{3}{2}m} \varepsilon^{1/2} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x^{-3m} \varepsilon + \begin{pmatrix} 0 & 1 \\ 0 & -m \end{pmatrix} x^{\frac{m}{2}-1} \varepsilon^{3/2} \right.$$

$$\left. + \ O(x^{-0m} \varepsilon^2) \right\} \tilde{u} \ .$$

The re-adjustment to integral exponents will be bypassed as inessential, since (11.16) can be diagonalized directly by the method of § 4. The result of this calculation is - details are omitted - that a transformation of the form

$$(11.17) \qquad \tilde{u} = \left[\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} + O(x^{-3m} \varepsilon) \right] \tilde{\tilde{u}}$$

changes (11.17) into a diagonalized system of the form

$$(11.18) \qquad \varepsilon^{3/2} x^{m/2} \frac{d\tilde{\tilde{u}}}{dx} = \left\{ \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x^{-3m/2} \varepsilon^{1/2} \right.$$

$$\left. + \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} x^{-3m} \varepsilon + O(x^{-7m/2} \varepsilon^{3/2}) \right\} \tilde{\tilde{u}} \ .$$

Now the diagonal system (11.18) and the scalar equation (11.14) must be integrated by the elementary procedure of § 9 and, finally, the several transformations performed in this section can be combined to yield the final result. Neither the details of the final formula nor the calculations have much intrinsic interest, except for the structure of

the exponent in (1.3), which is the decisive part of the asymptotic solution. For $m > 1$ that exponent is

$$\tilde{\varepsilon}^{-H} Q(x,\varepsilon) = \operatorname{diag}\{\varepsilon^{-3/2} \frac{i}{m/2-1} x^{-(m/2-1)}[1 + O(x^{-\frac{3}{2}m} \varepsilon^{1/2})] ;$$

$$- \varepsilon^{-3/2} \frac{i}{m/2-1} x^{-(m/2-1)}[1 + O(x^{-\frac{3}{2}m} \varepsilon^{1/2})] ;$$

$$- \varepsilon^{-2} \frac{x^{m+1}}{m+1} [1 + O(x^{-3m} \varepsilon)]\} .$$

Thus, in the notation of the general theory, $\tilde{\varepsilon} = \sqrt{\varepsilon}$, $\tilde{x} = x$ or $x^{1/2}$, depending on whether m is even or odd, and $H = 4$. The restraint index κ_1, is $1/3m$, in terms of x and ε.

The condition (11.3) is clearly not satisfied. It must be expected that this fact will make the central connection problem at the turning point harder than for problems where the solution remains bounded, as $\varepsilon \to 0$, in some parts of $\mathcal{D}_\varepsilon$.

The properties of the differential equation (11.4) in the shrinking disk around $x = 0$ are very complex. Details of a general analysis - though not a solution of the problems that arise - are given in [10] and [6], [7].

The natural tool is a "stretching" transformation

(11.19) $$x = \xi \varepsilon^{\rho}, \ \rho > 0 .$$

It changes (11.4) into

$$\varepsilon^{2-\rho}\frac{dy}{d\xi} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon & 0 & \xi^m \varepsilon^{m\rho} \end{pmatrix} y \,. \tag{II.20}$$

If $\rho < 1/3m$ any finite disk in the ξ-plane overlaps with $\mathcal{D}_\varepsilon$, at least for small enough ε. However, as one can check directly, the theory of this paper, when applied to (II.20) does not extend the validity of the asymptotic analysis to points closer to the origin than $O(x\varepsilon^{1/3m})$. For $\rho = 1/3m$, the shearing indicated by the theory is $y = \mathrm{diag}(1, \varepsilon^{1/3}, \varepsilon^{2/3})z$ and leads to

$$\varepsilon^{(5m-2)/3}\frac{dz}{d\xi} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & \xi^m \end{pmatrix} z \,.$$

Now, the origin is no longer a turning point, but there are $3m$ isolated points in the ξ-plane where some eigenvalues of the coefficient matrix coalesce. Thus, the local problem in the x-plane has become a very complex global problem in the ξ-plane.

The occurrence of such secondary turning points after the stretching is typical, whenever the original Puiseux polygon has more than one side. The paper [11] by M. Nakano and T. Nishimoto illustrates well these complications.

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Asymptotic expansions are calculated for systems of analytic linear ordinary differential equations near a pole with respect to a small parameter ε. If $x = 0$ is a turning point, no matter how complicated, the expansions are valid in domains which grow, as $\varepsilon \to 0+$, in such a way that their distance from $x = 0$ tends to zero. The paper considerably simplifies an earlier investigation by M. Iwano.

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